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**NEW GROWTH THEORY: AN EXPOSITORY
DEVICE**

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REVIEW

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New Growth Theory: An Expository Device

Preamble

Sudhindra Nath Datta, in dedicating the first ever collection of his poems¹ to Rabindra Nath Tagore, acknowledged a debt that he thought he was helpless to repay. I know of no better way to record my own indebtedness to Professor Mihir Rakshit than to recall that inscription. But for my close association with this great scholar, I would have been a destitute in the world of academics.

1 Introduction

A **Theory of Growth** attempts to explain the equilibrium growth rate of an economy's per capita output. Formal research on the determination of the equilibrium growth rate dates back to Roy Harrod (1939) and Evsey D. Domar (1946). This theory maintained that a rate of growth that is consistent with the forces of demand in a free enterprise economy is unlikely to match the rate of growth implied by supply side considerations. Hence, an equilibrium growth rate need not exist except by chance. Moreover, the nonexistence of an equilibrium will be characterized either by excessive fluctuations (Harrod), or by unrecoverable capital costs (Domar). Solow (1956, 1957) on the other hand, believed from empirical evidence that the rate of growth of per capita output for

¹Tōn-ni, M.C. Sarkar and Sons, Calcutta, *circa* 1930-31.

the US economy had a clearly discernible and stable trend.² Accordingly, he concluded that the Harrod-Domar prediction was unduly pessimistic and offered an alternative theory to theirs which allowed for a growth equilibrium to exist. He argued moreover that this equilibrium rate of growth (of an economy's per capita output) must equal the rate of technological progress.

In Solow (1956), however, technical progress was a nebulous entity, consisting of an *exogenously* specified growth in factor productivity, resulting from *unexplained* causes. That is, Solow's solution for the equilibrium value of the rate of growth of per capita output had two serious shortcomings. It was *uncontrollable* and it was *arbitrary*.

Solow (1960) attempted to overcome the feature of arbitrariness through the device of a vintage capital model, which visualized the very process of capital accumulation as a vehicle for technical progress. Productivity of capital goods was linked to their vintages, newly installed capital being more productive than old. Unfortunately however, the productivity differential of new machines over the old, i.e., the rate of technical change, was still exogenous to the model, and this, as Phelps (1962) demonstrated, continued to keep the equilibrium rate of growth tethered to an arbitrary parameter.

The second aspect, viz., uncontrollability, implied that neither private agents nor the state had a role to play in determining the rate of steady growth for an economy,³ an uneasy implication for policy makers in developing economies.

Both unsatisfactory features would be removed of course if the model were to be able to solve for the rate of growth

²See, for example, Denison (1974)

³Of course, the Solow model did imply that the rate of growth was susceptible to purely *temporary* manipulations.

without reference to the exogenous rate of technical progress or, alternatively, if the latter could be endogenised. Unfortunately, there was little success for more than two decades in the construction of such models, except for the brief attempts by Arrow (1962) and Drandakis and Phelps (1966). Thanks to Barro (1990), Lucas (1988), Rebelo (1991), Romer (1986, 1990) and others, however, there has been a recent spurt of interest in the subject in the shape of a **New Growth Theory** (NGT). This theory views the equilibrium rate of growth of an economy as a variable whose value is determined by the actions of agents constituting the system.

The present paper is a pedagogic attempt to summarize a few of the advances in this area. Its principal contribution is in the development of an elementary supply-demand framework for classifying the contributions, where the object of demand as well as supply is the *the rate of growth* itself. It is not too clear to me if this should be viewed as a totally original approach, but to the best my knowledge, the matter has not been presented so far in these precise terms. Since the literature on NGT is somewhat difficult for a beginner, it is hoped that the simplified approach will serve as a quick introduction to the subject. The paper is narrow in its coverage and many interesting contributions are left out of its purview. The reader is however urged to try and apply the framework of this paper to classify a few of these. Amongst others, a significant omission of this paper is the work on quality ladders due to Grossman and Helpman (1991).

Analytically speaking, the paper concentrates only on equilibrium states and not on behaviour out of equilibrium.⁴ Two separate issues come up in this context. First, there is

⁴Section 4.1.3 is the only exception to this statement.

a question of the stability of equilibrium. Apart from a brief summary of Harrod's discussion of the problem,⁵ the matter will be avoided. In any case, not much progress has taken place in this respect in NGT. The second question relates to the "convergence controversy". It is a live issue and much has been written on the topic. Given the limited scope of the paper as well as the restrictions of space, this problem will not be discussed either. This is a major omission and readers are referred to a satisfying treatment of the subject in Barro and Sala-i-Martin (1995).

The paper is divided into two parts. The first summarizes the salient issues of Old Growth Theory. This sets up the background as well as an analytical framework for the discussion in the second section which deals entirely with the NGT.

2 Old Growth Theory

2.1 Harrod

Harrod takes off from Keynesian macro-theory, which dealt with the equilibrium *level* of output and income $Y(t)$ for an economy at a given point of time t . By contrast, Harrod was concerned with the *growth rate* g of $Y(t)$ that kept the economy in Keynesian equilibrium at all t .

The condition for $Y(t)$ to be in Keynesian equilibrium at t is that

$$S(t) = sY(t) = I(t), \quad (1)$$

⁵Domar's model is substantively different from Harrod's. But vis-a-vis the NGT, its message is not much different. Hence, it is left out of the present discussion.

where s represents the (constant) marginal and average savings propensity of households, $S(t)$ their aggregate real savings at t and $I(t)$ the real investment demand by business. This condition is equivalent to the statement that $Y(t)$, if produced by the entrepreneurs and received by the households as income, will be matched by an equal level of demand. In other words, entrepreneurs and households would be satisfied respectively with the production and income level $Y(t)$ if and only if equation (1) is satisfied. In Keynesian terminology, the level of output and income satisfying (1) is known as *effective demand* [Keynes (1936), pp. 25]. By an *effective demand problem*, one usually refers to a shortfall of investment below savings at any given volume of output.

Harrod took his cue from this idea and suggested that a constant rate of growth g of $Y(t)$ would leave the entrepreneurs satisfied if, along the associated growth path of output, (1) holds at each t . The link between g and (1) is provided in his theory by the entrepreneurs' investment demand function. At all t , this depends on their expectation about the rate of change $\dot{Y}(t)$ ($= \frac{dY(t)}{dt}$) of $Y(t)$, i.e., the expected rate of change of aggregate demand for final goods and services. Choosing a particularly simple representation of this relationship, viz.,

$$I(t) = v \frac{dY(t)}{dt} = v\dot{Y}(t), \quad v \text{ constant}, \quad (2)$$

Harrod proved the following result:

PROPOSITION 1 A constant rate of growth g leaves all agents satisfied iff $g = \frac{s}{v}$.

Proof: Any constant growth rate g must equal

$$g = \frac{\frac{dY(t)}{dt}}{Y(t)} = \frac{\dot{Y}(t)}{Y(t)} \quad \forall t.$$

Suppose now that $g = \frac{s}{v}$. Then,

$$S(t) = sY(t) = v\dot{Y}(t) = I(t).$$

Hence, (1) is satisfied at all t , so that all agents are satisfied:

Conversely, suppose that g keeps all agents satisfied. Then, (1) holds at all t . Since g is the rate of growth of the economy, entrepreneurs must expect the rate of change of output at t to be

$$\dot{Y}(t) = gY(t) \quad \forall t.$$

Hence, using (2),

$$sY(t) = S(t) = I(t) = v[gY(t)] \quad \forall t, \quad (3)$$

or,

$$g = \frac{s}{v}.$$

Q.E.D.

Harrod referred to the rate of growth derived above as the *warranted* rate of growth, g^w .⁶ As opposed to g^w , the rate of growth of output that would fully utilize all resources

⁶In anticipation of what follows, this may be referred to as a *demand rate of growth*, since it is the rate at which output must grow in order that the demand for it is satisfied at all t .

is the natural rate of growth, g^n .⁷ If $g^n > g^w$, an effort to grow at the rate g^n will generate an inflationary spiral.⁸ A similar argument holds if $g^n < \frac{s}{v}$. In this case, the full employment growth path will lead to demand deficiency, and eventually, to unemployment. The important point in both cases is that market forces cannot be depended upon to arrest the destabilizing movements.

Given that the factors that determine g^n , are unrelated to the ones determining g^w , a discrepancy between the two rates is normally unavoidable. Thus, Harrod predicted that a free enterprise system is unlikely to be characterized by sustained growth with full employment and price stability.

2.2 Solow

Solow's [Solow (1956)] interest revolves around the growth rate attainable *in the absence of an effective demand problem*. In order to abstract from the effective demand problem, he adopts a Walrasian rather than a Keynesian view of the macro-economy. There are only two perfectly substitutable

⁷This is the counterpart of the *supply rate of growth* discussed below.

⁸Thus,

$$g^n = \frac{\dot{Y}(t)}{Y(t)} > \frac{s}{v},$$

implies that

$$I(t) = v\dot{Y}(t) > sY(t) = S(t).$$

In other words, an excess of g^n over g^w leads to a state of excess demand for final goods and services. Since excess demand is a signal for buoyant markets, entrepreneurs would conclude that the rate of growth g^n is too small and attempt to grow at an even higher rate. Since g^n is already the full employment rate, however, this will lead to inflation.

factors of production (capital and labour), both augmentable over time. The household sector owns both these and derives its income by supplying their services inelastically to the business sector in perfectly competitive markets. The demand for factor services derives from profit maximization by the firms. The endowment of labour at each instant of time is a demographic parameter, while savings, i.e., the part of final output not demanded as consumer goods, are invested, i.e., used to augment the household's existing endowment of capital. Thus, as opposed to the Keynesian approach, *it is the household that undertakes both saving and investment decisions*. With a flexible wage-rental ratio, there is full employment of both factors, irrespective of the size of factor endowments. Since, by assumption, all output is demanded, either as consumption or as investment goods, the business sector is able to produce and sell *any* level of output, in particular the full employment level, and the effective demand problem does not arise.

Given this analytical abstraction, Solow goes on to solve for the equilibrium level of *steady state growth** for a competitive economy. The concept of steady state growth is defined as follows:

Definition of Steady State Growth

Steady State Growth is a dynamic state of the economy in which output (Y), capital (K) and consumption (C) grow at constant rates.

In what follows, the Solow equilibrium will be represented as a state of balance between demand and supply forces, where the object of demand as well as supply is "a steady state growth rate" of per capita consumption.

It is helpful to begin with the notion of demand. Given that demand and supply are equal at all t , the household sector's decision consists of the allocation of aggregate output or income at each point of time between consumption and savings. As already noted, the household saves *in order to* accumulate physical capital. The latter activity is geared towards achieving a *desired* per capita consumption flow *over time*. The household's consumption-savings decision at each t boils down therefore to the choice of a *constant* rate of growth of per capita consumption. This rate is the *demand rate of growth*. Thus, while the Keynesian effective demand problem is assumed away by Solow, the notion of demand is not altogether absent in his framework. Demand for the "level" is replaced by that for a "rate of growth". The demand rate will be seen to be a function $g^d(r)$ of the real rate of interest r . Hence, it is a demand function for a rate of growth.

Solow (1956) had assumed a constant savings rate, and hence, an ad hoc solution to the household sector's decision problem. Modern treatments of the subject however assume [in response to the dynamic optimization frameworks of Cass (1965) and Koopmans (1965)⁹] the problem to be solved *optimally* by an infinitely lived representative household possessing perfect foresight. Given the initial capital stock (K_0) and labour force (L_0), it chooses an optimal *path* of consumption (and hence savings) over infinite time. Typically, it is assumed to maximize the intertemporal welfare function given by

$$\int_0^{\infty} e^{-\rho t} L(t) \frac{1}{1-\sigma} (c(t)^{1-\sigma} - 1) dt \quad (4)$$

⁹Strictly speaking, Ramsey (1928) had anticipated them.

where $\frac{1}{1-\sigma} (c(t)^{1-\sigma} - 1)$ is the utility from *per capita* consumption $c(t) = \frac{C(t)}{L(t)}$ at t , $\rho > 0$ the discount parameter and $\sigma > 0$ the constant elasticity of marginal utility from consumption.¹⁰ The household's budget at each instant is given by

$$L(t)c(t) + \dot{K}(t) = r(t)K(t) + w(t)L(t) \quad (5)$$

where $\dot{K}(t)$ is aggregate investment (which equals aggregate household savings, for reasons outlined above), $r(t)$ the real rate of interest and $w(t)$ the wage rate at t .¹¹ The assumption of perfectly competitive markets implies that $r(t)$ and $w(t)$ are parametrically given to the household.

The household's choice of the demand rate is given by the following result:

PROPOSITION 2 *The demand rate of growth of per capita consumption is a monotone increasing, linear function of r . Specifically, it has the form $g^d = g^d(r) = \frac{r-\rho}{\sigma}$.*

For an informal proof of the proposition,¹² assume that $c^*(t)$ is the optimal path of per capita consumption and g^d its constant rate of growth. Suppose the household decides to reduce consumption by a small unit at t_0 , invest the amount at the going rate of interest r and consume the entire proceeds in period t_1 . Along the optimal path, this perturbation should leave the household indifferent. Thus, in terms of

¹⁰A utility function involving variable elasticity of marginal utility raises problems for the existence of a steady state equilibrium growth rate in the presence of technical change.

¹¹Without loss of generality, it will be assumed throughout that there is no physical depreciation of capital.

¹²A more formal proof is relegated to the appendix.

marginal gains and losses, $c^*(t)^{-\sigma} = e^{(r-\rho)(t_1-t_0)} c^*(t_1)^{-\sigma} = e^{(r-\rho-\sigma g^d)(t_1-t_0)} c^*(t_0)^{-\sigma}$. Solving, the result follows.

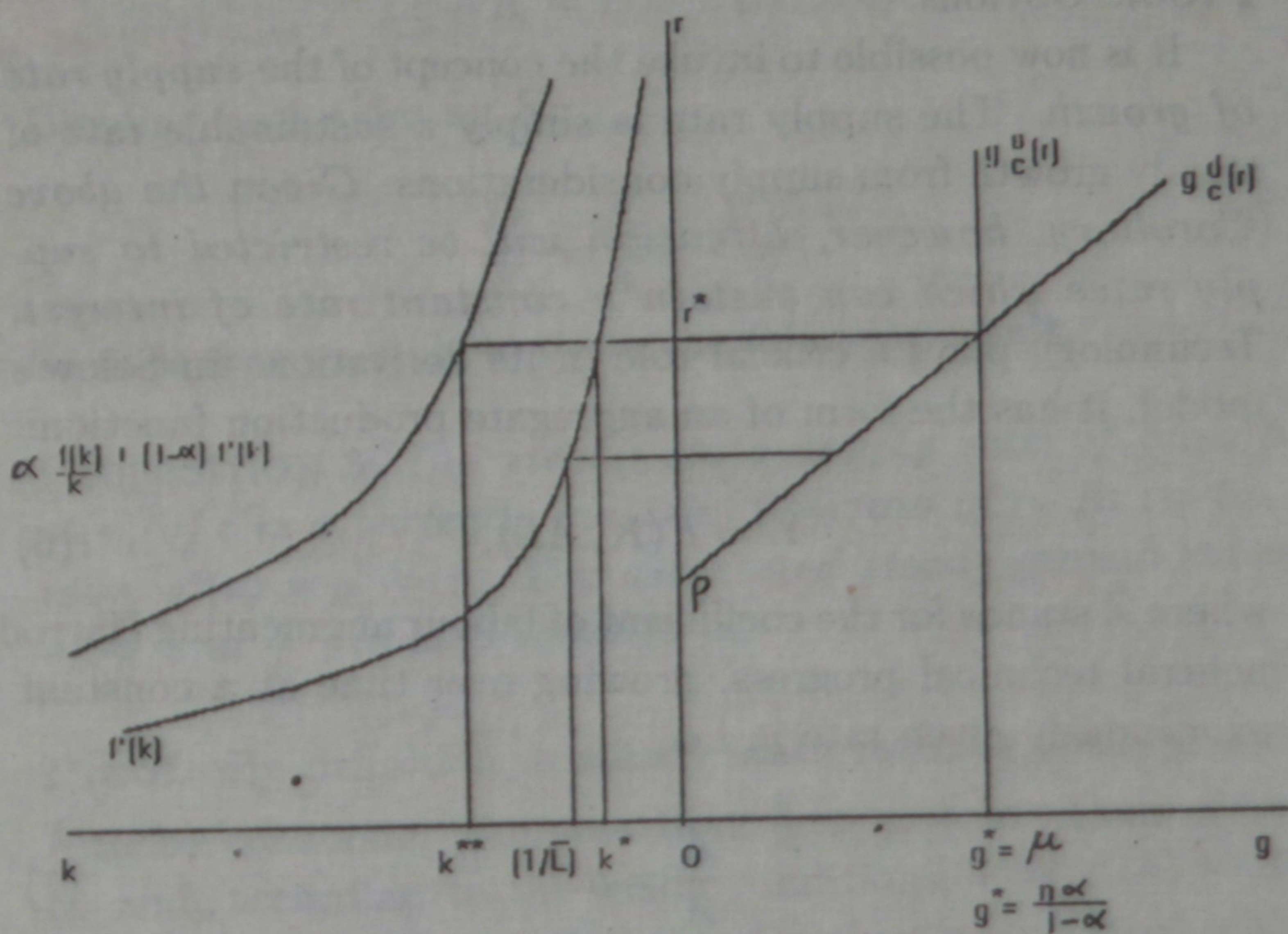


Figure 1: SOLOW, ARROW, D'AUTUME-MICHEL

On the right hand panel of Figure 1, the curve $g^d(r)$ shows the demand rate of growth as a function of the rate of interest. If a steady state equilibrium rate of growth of c is to represent a balance between demand and supply forces, it must be consistent with the function $g^d(r)$. In other words, it must lead to a (g^d, r) pair located on this curve. A necessary implication of this is the following Corollary to Proposition 2.

COROLLARY 1 *The real rate of interest associated with the equilibrium steady state growth path is a constant.*

Proof. Obvious.

It is now possible to invoke the concept of the *supply rate of growth*. The supply rate is simply a sustainable rate of steady growth from supply considerations. *Given the above Corollary, however, attention will be restricted to supply rates which can sustain a constant rate of interest.* Technology plays a crucial role in its derivation. In Solow's model, it has the form of an aggregate production function:

$$Y = F(K, AL), \quad (6)$$

where A stands for the coefficient of labour augmenting Harrod-neutral technical progress, growing over time at a constant exogenously given rate μ , i.e.,

$$\frac{\dot{A}}{A} = \mu. \quad (7)$$

Similarly, the labour force $L(t)$ is assumed to grow exponentially at the rate n . Thus, (6) can be represented in per capita terms as

$$y = f(k) \quad (8)$$

with $y = \frac{Y}{AL}$ and $k = \frac{K}{AL}$. The assumptions on F guarantee that $f'(k) > 0$ and $f''(k) < 0$.¹³ Since both factors are fully

¹³Further restrictions need to be imposed on f in the form of the so called Inada conditions. These are $f'(k) \rightarrow 0$ as $k \rightarrow \infty$ and $f'(k) \rightarrow \infty$ as $k \rightarrow 0$. The conditions imply unlimited diminishing returns with respect to each factor.

employed and there is no effective demand problem, it follows that

$$Lc + \dot{K} = Y = F(K, AL).$$

Dividing both sides by K ,

$$\frac{Lc}{K} + \frac{\dot{K}}{K} = \frac{f(k)}{k}. \quad (9)$$

It is now possible to prove the following result.

PROPOSITION 3 *The steady state supply rate of growth $g^s(r)$ of c is a perfectly inelastic function of r . In particular, $g^s(r) = \mu \forall r$. The associated steady growth rates of $\frac{K}{L}$ and $\frac{Y}{L}$ are also equal to μ .*

Proof. By definition, a steady state requires both $\frac{\dot{K}}{K}$ and $\frac{\dot{c}}{c}$ to be constants. Suppose that $\frac{\dot{K}}{K} > \mu + n$. Then, $k \rightarrow \infty$ and, according to the Inada conditions, $r = f'(k) \rightarrow 0$ monotonically. Hence, the rate of interest r associated with the steady state path is not a constant.

Now suppose that $\frac{\dot{K}}{K} < \mu + n$. Then, $k \rightarrow 0$. The Inada conditions imply that $r = f'(k) \rightarrow \infty$, so that r is once again not a constant.

Thus, the only possible supply rate of growth of K is $\mu + n$. When K grows at this rate, k is a constant. Hence, $r = f'(k)$ is a constant also.

Going over now to (9), it is easy to see that if K grows as $\mu + n$, then so does $C = Lc$. Alternatively, $\frac{K}{L}$ and c grow at the rate μ .

Further, when c and $\frac{K}{L}$ grow as μ , so does $\frac{Y}{L}$. This is seen by a total differentiation of (6) which yields

$$\begin{aligned}\frac{\dot{Y}}{Y} &= \pi \frac{\dot{K}}{K} + (1 - \pi) \frac{\dot{AL}}{AL} \\ &= \pi \frac{\dot{K}}{K} + (1 - \pi)(\mu + n),\end{aligned}\quad (10)$$

where π and $1 - \pi$ are respectively the shares of capital and labour in Y . Since π is a positive fraction, the result follows.

Q.E.D.

Whereas the rate of growth μ of c implies a constant value of k , the level of the constant is not determinate. In other words, the same rate of growth μ is realisable for any value of $k \in (0, \infty)$ so long as the chosen k is kept fixed. Since competitive equilibrium gives $r = f'(k)$, however, this means that the value of r associated with μ is also indeterminate. Hence, the supply rate curve is a perfectly inelastic function of r in the $r - g^s$ plane. As noted earlier, a growth equilibrium obtains when the demand and the supply rates are equal. This gives:

PROPOSITION 4 *The equilibrium rate of steady state growth of c , $\frac{K}{L}$ and Y are given by $g^* = \mu$. Further, it is associated with a constant rate of interest r^* , a constant $k = k^*$ and a constant savings rate $s = s^*$ such that*

$$\begin{aligned}g^* &= \frac{r - \rho}{\sigma} = \mu, \\ f'(k^*) &= r^* = \rho + \mu\sigma\end{aligned}\quad (11)$$

and

$$\mu + n = \frac{\dot{K}}{K} = \frac{s^* f(k^*)}{k^*}.$$

Proof. The value of g^* follows trivially. The implications for r and k follow from Proposition 2 and profit maximization. The value of s^* is definitional.

Q.E.D.

The growth equilibrium is visualized in Figure 1 as an intersection of the two curves, $g^d(r)$ and $g^s(r)$.¹⁴ The unique solution to (11) is referred to as the *modified golden rule* value of k .¹⁵ The determination of k^* is shown on the left hand panel of Figure 1.

Analytically speaking, the economy is visualized to be at k^* from the very beginning, no questions being asked as to how it ever went there. Given $L(0)$ this constrains the initial value of $K(0)$ to be equal $k^* L(0)$. Thus, it is being *assumed* that the economy has the required capital stock to ensure an instantaneous adjustment to the steady state path.

Even though the equilibrium rate of growth is viewed as an intersection of a demand and a supply curve, it should be obvious that it is the supply side, as brought out by Proposition 3, which dominates. For example, a change in the

¹⁴In terms of the right hand panel of Figure 1, Harrod's theory reduces to the argument that both g^w and g^n are perfectly inelastic functions of r with no point in common.

¹⁵The qualifier "modified" distinguishes the present exercise from the original formulation of the problem by Phelps (1961, 1965) in terms of an *undiscounted* welfare function.

parameters of demand, i.e., shifts in the demand curve, will leave the equilibrium rate of growth unaffected. An increase in thriftiness, caused, say, by a tax on consumption, will fail to affect the equilibrium rate of growth of the system.

The reason underlying the result is to be found in the operation of the law of diminishing returns to the endogenously accumulated factor, viz., capital K , in the face of the exogenously growing factor AL , viz., efficient labour. It is not possible for the *constant* growth rate of capital in steady state to be either above or below the *given* constant rate of growth of efficient labour. For, diminishing returns would lower or raise the marginal productivity of capital, and hence the rate of interest. The latter in turn will make it impossible for the rate of growth of consumption to remain steady. Steady state equilibrium occurs when the two factors grow at the same rate. Since $\mu + n$ is rigidly specified, however, this can be the only equilibrium rate of growth of capital also. Moreover, given that capital has only one possible equilibrium rate of growth, all other rates of growth adjust to it.

The rigidity of the growth rate would clearly disappear in the absence of diminishing returns, for in that case, the rate of growth of capital can be *chosen* by the model, independent of exogenous considerations.¹⁶ Alternatively, even in the presence of strong diminishing returns, the equilibrium growth rate can be endogenously chosen if μ or n were

¹⁶Rebelo (1991) offers important insight in this respect. However, the absence of diminishing returns is not essential for the delinking of the growth rate of capital from $\mu + n$. Indeed, if $\frac{f(k)}{k} \rightarrow \gamma$ stays bounded away from zero as $k \rightarrow \infty$, i.e., one of the Inada conditions fails and the diminishing returns are weak, the equilibrium growth rate *may* no longer adjust to $\mu + n$.

themselves to be accorded the status of controllable rates of growth. One way would be to offer an interesting *theory* of technical change.¹⁷ Another route is to go for a theory of fertility, so that the rate of population growth is determined by the model.¹⁸ A third alternative is to replace labour by human capital and allow the latter to be endogenously accumulable (through schooling, for example).¹⁹ The rest of the paper reports on a few of these developments.

3 New Growth Theory

3.1 Arrow: A Theory of Technical Change

Solow (1960) represents one of the earliest attempts to theorize about technical progress by linking it to the process of capital accumulation. New investment is seen to be *embodied* in equipment having higher productivity than those installed earlier. The approach has both shortcomings of Solow (1956). First, the rate of increase in productivity being exogenous to the system, the equilibrium growth rate is still supply determined and uncontrollable. Secondly, the source of technical progress is remains unexplained.

Arrow (1962) takes a significant step towards offering a theory of productivity growth, thereby endogenising the equilibrium rate of technical progress for the economy. As far as this theory goes, he attributes productivity increases over time to learning, i.e., the accumulation of experience, on the part of the labour force. Experience is gathered in workshops while producing output with the help of machinery and

¹⁷Arrow (1962), Romer (1990).

¹⁸Becker and Barro (1988).

¹⁹Uzawa (1965), Lucas(1988), Rebelo (1991).

equipment. Hence, technical progress amounts to "learning by doing". Every new piece of equipment has to be "broken in" so to speak, thus creating room for learning.²⁰ Since it is in the course of machinery handling that learning takes place, Arrow measures the index of experience by the value of cumulative *gross* investment (CIG) at any point of time.²¹ As he viewed it, an increase in investment today raises the size of CIG from tomorrow onwards above what it would otherwise be, thus bequeathing a more experienced and productive labour force to the future. In other words, technical change is in the nature of an intertemporal externality generated by the process of capital accumulation. As such, it has an *embodied* form, later machines are more productive than earlier ones.

In what follows, however, a *disembodied* version of the Arrow exercise [suggested by Sheshinski (1967)] will be considered. Technical change is still an externality, but it is atemporal. An investment activity, irrespective of the firm in which it is located, adds to the CIG for the entire economy, and the benefit of increased labour productivity in the initiating firm *spills over* to *all* coexisting firms.²² Thus, labour productivity goes up for all existing machinery, irrespective of their vintages.

Formally speaking, the only change this introduces in the

²⁰In Arrow's words: "Each new machine put to use changes the production environment, thereby inducing the workers to learn."

²¹Thus, in the presence of physical depreciation of capital, experience is measured by the aggregate capital stock that *would be* in existence in its absence. Arrow also offers clear reasons as to why cumulative gross output is not a satisfactory index of learning.

²²In concrete terms, it might help to think of labourers from different organizations converging to the local pub as it were, where they get a chance to exchange information.

Solow (1956) model is in the specification of $A(t)$, which is now a function of CIG at t . In the absence of physical depreciation, the latter is equal to the aggregate capital stock $K(t)$. The exact functional form assumed by Arrow and Sheshinski is

$$A(t) = K(t)^\alpha, \quad \alpha > 0. \quad (12)$$

Capital accumulation has diminishing, constant or increasing productivity in the learning activity depending on whether $\alpha < 1, = 1$ or, > 1 .

The technology is therefore given by

$$Y = F(K, K^\alpha L), \quad (13)$$

where $F_{11} < 0$ and $F_{22} < 0$, so that there are diminishing returns to K (if the spillover in the form of K^α is ignored) and $K^\alpha L$ as a whole. Also, the function F displays constant returns to scale in K and $K^\alpha L$.²³ But, the returns to scale in K and L are increasing. The increasing returns being external to the firm, however, the equilibrium for the system is still sustainable by Solow type competitive markets.

The interesting change now, as compared to the Solow formulation, is that the *social* marginal productivity of capital $\frac{\partial F}{\partial K} = \alpha \frac{f(k)}{k} + (1 - \alpha)f'(k)$, is strictly greater than its *private* marginal product $F_1 = f'(k)$. A private producer, in computing the profitability of an additional dose of capital, will be ignoring the gain it generates for other producers.²⁴ Also,

²³As before, the function may be expressed in the form (8), with no change in the definitions of y and k . The properties of $f(k)$ remain unaltered too vis-a-vis the variable k .

²⁴Intuitively speaking, the productive sector may be imagined to be made up of M identical firms, each producing $F(\frac{K}{M}, K^\alpha \frac{L}{M})$. Under con-

while the private marginal productivity of capital is diminishing (i.e., $F_{11} < 0$), this may not be the case for its social marginal productivity. This is evident from the expression for average social productivity of capital

$$\frac{Y}{K} = F(1, K^{\alpha-1}L) \quad (14)$$

which decreases, remains constant or increases according as $\alpha <, =$ or, > 1 . When $\alpha < 1$, the diminishing social marginal productivity of K is traceable (as in Solow) to the existence of the exogenously accumulating factor labour.²⁵ In this case, the equilibrium rate of growth is strongly influenced by the rate of population growth, though the expression for it is less obvious than in Solow. In fact, the equilibrium rate of growth of efficient labour is now a blend as it were of an endogenous factor, viz., the rate of growth of learning, which is a function of the rate of capital accumulation, and the exogenous population growth rate. Thus, while the rate of capital growth is determined by the rate of growth of efficient labour, the latter is itself influenced by the former. As a result, the equilibrium rates of growth of capital and efficient labour are solutions to a pair of simultaneous equations and hence, endogenously determined. However, to the extent that the exogenous rate of population growth continues to influence the steady state growth rate, the case $\alpha < 1$ can only be a partial solution to the Solow problem. This shows up in the

stant returns, this means that aggregate production is given by (13). An increase in $\frac{K}{M}$ in all firms brings about a corresponding rise in K , but each firm, being negligibly small, is aware only of the change in its own capital stock. Hence, it does not take into account the effect of the social marginal productivity of the change in the private capital stock.

²⁵In other words, the increasing returns to K and L is not strong enough to counter the dampening effect of an exogenously growing labour force.

fact that while the rate of growth is no longer unexplained, it is still uncontrollable.

As can be easily surmised, however, the exogenous rate of population growth ceases to play a role when $\alpha \geq 1$, for now there are no diminishing returns to capital accumulation. Full-fledged theories of endogenous growth result in the process.

The discussion that follows will be divided up into three parts, corresponding to $\alpha < 1$, $= 1$ or, > 1 .

3.1.1 $\alpha < 1$

In Solow, the position of g^s is determined by μ , while μ is itself exogenously specified. The important progress made by Arrow's theory of technical change is to set up a simultaneous equation system whose solution determines the position of g^s . Figure 1 represents Arrow's equilibrium also, but the value of μ is now specifiable in terms of other parameters of the model. This is the content of the next result.

PROPOSITION 5 *The equilibrium rate of growth for an economy with learning is given by $\mu = \frac{n\alpha}{1-\alpha}$.*

Proof. The principle of determination of equilibrium growth rate is the same as in Proposition 4. The only difference lies in the determination of the supply rate of growth, which is now endogenous. Proposition 3 and equation (12) imply that

$$\frac{\dot{K}}{K} = \mu + n$$

and

$$\mu = \alpha \frac{\dot{K}}{K}$$

Solving these simultaneous equations in μ and $\frac{\dot{K}}{K}$,

$$\begin{aligned} \mu = \alpha \frac{\dot{K}}{K} &= \frac{n\alpha}{1-\alpha} \\ \frac{\dot{K}}{K} &= \frac{n}{1-\alpha} \end{aligned} \quad (15)$$

Q.E.D.

The unique solution value $\frac{n\alpha}{1-\alpha}$ is determined by the principle that decreasing (increasing) returns operate when per capita capital grows at a higher (lower) rate. This makes $\frac{n\alpha}{1-\alpha}$ the only maintainable rate of growth of $\frac{K}{L}$.

Although Arrow had a concrete theory of technical progress, his conclusions were nevertheless disappointing insofar as they failed to suggest how policy prescriptions or perturbations in preference parameters might influence the *growth rate*.

There is a role for policy in Arrow's model however, even if the *rate of growth* cannot be affected. As already noted, private entrepreneurs tend to underestimate the social marginal productivity of capital by its *private* marginal product. It is the latter and not the former that is equated to the real rate of interest for profit maximization. This leads to a suboptimal allocation of resources and the competitive equilibrium fails to be a social optimum. To appreciate the nature of market failure implied by the Arrow model, the equilibrium of the private enterprise economy needs to be compared with that

of a *command* economy where the allocation of resources is carried out by an omniscient planner. The latter is endowed with exactly the same welfare function (4) as the representative household, but unlike private agents, takes into account the social productivity of capital. Although this has no effect on the equilibrium rate of growth, it shows up as a smaller modified golden rule value of k in a market equilibrium as compared to the social optimum. This is the content of the next result.

PROPOSITION 6 *The equilibrium value k^* of k in a competitive economy is strictly less than its equilibrium value k^{**} in a command economy.*

As in the case of Proposition 2, the formal proof of this proposition is left for the Appendix. Informally, the argument rests on the observation that the command economy replaces the rate of interest r in Proposition 2 by the social marginal productivity of capital $\alpha \frac{f(k)}{k} + (1 - \alpha) f'(k)$. Equating this to the expression for g^s given by Proposition 5, k^{**} satisfies

$$\left[\alpha \frac{f(k^{**})}{k^{**}} + (1 - \alpha) f'(k^{**}) \right] = \rho + \frac{n\alpha}{1 - \alpha} \sigma.$$

On the other hand, the market economy equates r to $f'(k)$ with the result that the same sequence of arguments gives

$$f'(k^*) = \rho + \frac{n\alpha}{1 - \alpha} \sigma.$$

The proposition now follows from the strict concavity of $f(k)$.

As already noted, Proposition 6 creates room for policy intervention to improve upon the competitive solution. Since the increase in wages resulting from improved labour productivity is traceable to capital which receives less than its true

marginal product in a competitive equilibrium, the obvious policy implication is to tax wage income (which leaves the inelastic labour supply unaffected) and subsidise capital.

Apart from the equilibrium rate of growth turning out to be impervious to policy manipulations, a second major shortcoming of the Arrow theory is that technical progress, though endogenously generated, is viewed as an inevitable (or, unavoidable) *byproduct* of the process of capital accumulation. It is not linked to the *deliberate* actions of economic agents who are known in real life to innovate in search of higher profits from production.

The major difficulty of treating technical change as a choice variable of the model stems from the fact that it is not a marketable commodity. It is, in fact, both nonrival as well as nonexcludable, i.e., it is a pure public good. For example, in the Arrow-Sheshinski model, it is nonrival, since all firms enjoy it simultaneously; and it is nonexcludable, since it has the form of an external effect. This is an important issue addressed by Romer (1990).

3.1.2 $\alpha = 1$

Arrow did not consider this case in depth, but it has a rather interesting feature as a recent study by d'Autume and Michel (1993) indicates. For this specification of the parameter α , the Arrow production function is globally linear in capital. As a result, it has two of the features (viz., linearity and a theory of technical change) outlined at the end of Section 2 which can lead to an endogenous determination of the equilibrium rate of growth.²⁶ For the sake of continuity, the neoclassical

²⁶It goes without saying that any one of these features would have been sufficient. Of course, in Arrow they are not separable from each other.

framework will be maintained, although the original results were presented in Arrow's embodied form.

Following the steps of Proposition 3, it must still be true that

$$\begin{aligned}\frac{\dot{K}}{K} &= \mu + n \\ &= \frac{\dot{K}}{K} + n, \quad \alpha = 1.\end{aligned}\tag{16}$$

However, this can happen only if $n = 0$. Thus, in the present model, a necessary condition for the existence of an equilibrium growth rate is a *stationary* level of population, i.e., L must be a constant, say \bar{L} . As opposed to Proposition 5, capital K and efficiency labour $K\bar{L}$ must have no choice but to grow at a common rate in this model and diminishing returns does not have a chance to operate. Thus, there is no supply mechanism to nail down the equilibrium rate of growth. In fact, the supply rate curve is completely missing. The equilibrium rate of growth is nevertheless determinate, since (16) implies a stationary value for k , which, by definition, equals $k^* = 1/\bar{L}$. Consequently, (11) admits a unique profit maximizing value of r , viz., $r^* = f'(1/\bar{L})$. Finally, Proposition 2 determines $g^* = \frac{r^* - \rho}{\sigma}$. As can be seen, a change in ρ and σ would now have an effect on the growth rate. So would policy, such as a tax on interest income. The conclusions are summarized below.

PROPOSITION 7 *When $\alpha = 1$, equilibrium growth implies $L = \bar{L} = \text{constant}$. Further, the rate of equilibrium growth of c , $\frac{K}{L}$ and $\frac{Y}{L}$ is $\frac{r^* - \rho}{\sigma}$, where $r^* = f^{-1}(\frac{1}{\bar{L}})$. A change in preference parameters affects the growth rate,*

while a change in the size of the labour force affects both the growth rate (through r^*) as well as the modified golden rule value of k .

It may be noted that while the equilibrium solution is no longer tied to the rate of population growth, it does still depend on the size of the labour force, since the latter determines the equilibrium rate of interest and hence the equilibrium growth rate via Proposition 2. Figure 1, demonstrates the nature of this *demand determined* equilibrium.

Once again, the command economy takes into account the externality factor in its optimality calculation and this shows up in the form of a higher value accorded to the social marginal productivity of capital as compared to the market economy. Since the production function can be rewritten as $K\bar{L}f(1/\bar{L})$, the social marginal productivity is a constant $\bar{L}f(1/\bar{L})$ and equals the social average productivity $\frac{f(k)}{k}$. As a result, the equilibrium rate of growth for the command economy is

$$g^{**} = \frac{\bar{L}f(1/\bar{L}) - \rho}{\sigma}.$$

Hence, the following proposition emerges:

PROPOSITION 8 *The equilibrium value g^* of the rate of growth in a market economy is strictly less than its equilibrium value g^{**} in a command economy.*

As opposed to the case $\alpha < 1$, it is now the rate of growth that is higher in the command economy and *not* the equilibrium value of k .

In the d'Autume and Michel work, the rate of population growth takes a back seat in the explanation of the rate of

growth of the economy. This feature will be shared by the remaining models to be discussed also, though, as opposed to d'Autume and Michel, they do not have a *logical* need to assume the rate to be zero. *Nevertheless, since the exogenous rate of population growth has no analytical significance in the next few models, it will henceforth be assumed to be identically zero without any loss of generality.*

3.1.3 $\alpha > 1$

Given the objective of the paper outlined in the Introduction, this case represents a detour. First, it does not permit the system to be in steady equilibrium. The steady state is where the economy tends over time. As such, the entire analysis is restricted to out of steady state behaviour. Secondly, the model is an effort to explain why poor as well as rich countries may display similar rates of growth, a phenomenon that the simple Solow model would not predict, since diminishing returns (or, capital saturation) should imply a lower out of steady state growth rate for economies with a higher per capita output. This part of the discussion falls within the purview of the "convergence question".

Both issues are beyond the scope of the paper. Nevertheless, a brief treatment is offered to round up the issues raised by the Arrow formulation of technical progress. It must be remembered of course that Arrow had ignored the case $\alpha > 1$ altogether. On the other hand, it is Romer's [Romer (1986)] earliest attempt to tackle the Solow problem. Romer calls for a distinction between physical and knowledge capital. Unlike Arrow, it is the accumulation of knowledge rather than physical capital that contributes to an increase in labour productivity. In the changed scenario, *I* stands for

physical investment and K for the stock of knowledge capital, though, at this stage of Romer's work, the precise nature of knowledge capital is left somewhat vague. As before, savings being equal to investment,

$$I = F(K, K^\alpha L) - C.$$

On the other hand, "new" knowledge capital (i.e., \dot{K}) is produced through the application of physical investment and the *entire* stock of knowledge capital by means of a linear homogeneous production function $G(I, K)$. In other words,

$$\dot{K} = G(I, K) = Kg\left(\frac{I}{K}\right). \quad (17)$$

where $g\left(\frac{I}{K}\right) = G\left(\frac{I}{K}, 1\right)$.

The fact that the entire quantum of K is present simultaneously in both F and G indicates that it is being viewed as a nonrival good.²⁷ Amongst other things, the existence of such a factor calls for a careful consideration of the market structure, for it is not entirely clear that a competitive market structure can sustain the growth path any longer. Indeed, if the final output [i.e., $F(., .)$] is produced in a competitive market, then K and $K^\alpha L$ are paid their respective marginal products (i.e., F_i , $i = 1, 2$) and this exhausts the total output. However, in that case, K is not a paid factor in the production of "new" knowledge. A plausible story may be that the production of $C + I$ is a market oriented activity, while accumulation of K is an activity internal to the household. The latter involves the nonrival input K as well as the *productive* consumption of the rival good I .

²⁷The conceptual implication of this is brought out in great detail in Romer (1990), though this paper does not deal with a generalization of Arrow.

The rate of growth of efficiency labour is

$$\frac{\dot{K}}{K} - \alpha \frac{\dot{K}}{K} < 0$$

Hence, $k \rightarrow 0$ and $\frac{f(k)}{k} \rightarrow \infty$. Writing (17) as

$$\frac{\dot{K}}{K} = g\left(\frac{sf(k)}{k}\right),$$

it follows that $\frac{sf(k)}{k} \rightarrow \infty$ if s is bounded away from zero. Romer assumes that

$$g(\cdot) \leq \bar{g}, \quad \text{an exogenously specified bound.}$$

The rate of growth of knowledge capital is bounded above by \bar{g} however high may be the ratio $\frac{I}{K}$. As a result, (9) implies that $\frac{C}{K}$ diverges as $\frac{\dot{K}}{K}$ approaches \bar{g} . Similarly, the rate of growth of efficiency labour approaches $\alpha\bar{g}$ and $\frac{\dot{Y}}{Y}$ lies trapped over time between $\pi\frac{\dot{K}}{K} + (1-\pi)\frac{\dot{K}^\alpha L}{K^\alpha L}$. To summarize:

PROPOSITION 9 *When $\alpha = 1$, the economy is never in steady state equilibrium. Under the assumption that the rate of growth of knowledge capital is a bounded function of the ratio of physical investment to knowledge capital, K and efficiency labour $K^\alpha L$ grow at constant, though not equal rates.*

It goes without saying that there would once again be a difference between the market and the command economy and this would show up in the *paths* of capital accumulation.

As already noted, the model provides an explanation for the often observed phenomenon that a rich and a poor economy exhibit the same rate of growth in spite of wide divergences between the levels of per capita variables. In fact,

the increasing returns characterizing the model implies that initial discrepancies in per capita magnitudes get magnified over time at the same time that the rates of growth converge. Thus,

$$K_l(t) = K_l(0)e^{g_l(t)}$$

and

$$K_s(t) = K_s(0)e^{g_s(t)},$$

where l and s stand respectively for the large and small countries. Writing $K_l(0) - K_s(0) = \epsilon > 0$, it is easy to show that $K_l(t) - K_s(t) \rightarrow \infty$ when $g_l(t) - g_s(t) \rightarrow 0$.

3.2 Rebelo: The Role of a Linear Technology

The d'Autume-Michel model can be viewed as a special instance of a general insight provided by Rebelo (1991) that the equilibrium rate of growth stays delinked from exogenous factors even in the presence of diminishing returns, so long as the latter is restricted to sectors other than the one producing capital goods.

To be precise, Rebelo divides up the factors of production into a reproducible and a nonreproducible²⁸ group, say, K and L . The former is a composite of physical and human capital, while the latter may be called land. Growth in the labour force is now replaced by a growth in human capital, but the

²⁸As will be seen below, it is not so much the nonreproducibility of the factors as the exogeneity of the rules governing their dynamics that is analytically important.

latter is endogenous.²⁹ There are two sectors producing respectively an investment good (I) and a consumption good (C). In the absence of labour growth, per capita consumption c is identical with C . The investment good is produced by a non-sector specific (though rival) capital good (K) using a linear technology

$$I = A(1 - \phi)K, \quad (18)$$

where A is a technological constant, ϕ the endogenously determined share of existing capital used in producing I and, by definition, $I = \dot{K}$. The consumption good is produced by a Cobb-Douglas technology using K and the nonreproducible factor L :

$$C = B(\phi K)^\alpha L^{1-\alpha}, \quad (19)$$

where B is a constant.

A decision to save is a decision to abstain from consumption, and this, in a standard one sector model, amounts to a simple one-to-one diversion of a part of the output towards capital formation. In the present two sector framework, the consumption-savings choice is a more complicated problem and amounts to a decision to *produce* less of C by diverting a larger fraction of K to the production of I . Keeping this interpretation of savings in mind, the demand rate of growth g_c^d of C is given by Proposition 2, with r replaced by r_c , the real rate of interest in units of the consumption good. Thus,

²⁹In the linear version of the model considered by Rebelo, human and physical capital are inseparable. A separation is possible in his nonlinear version of the problem. However, this part of the Rebelo exercise will be discussed in the next section as an extension of Lucas (1988), since the source of growth is not traceable here to an absence of diminishing returns.

the equilibrium rate of growth is known once the equilibrium value of r_c is solved for. In the absence of (19), i.e., if the model had a single sector with (18) as the technology, profit maximizing entrepreneurs would equate the real rate of interest to the marginal product A of capital. Hence, the equilibrium rate of growth would be known as in d'Autume and Michel. On the other hand, if (19) represented an *aggregate* production function as in Solow (1956), the rate of capital accumulation would be affected by diminishing returns and, in the presence of the Inada conditions, the equilibrium rate of growth of capital and output equal zero, since L is not growing by assumption.

In Rebelo's case, however, the investment sector represents an indecomposable sub-sector of the economy.³⁰ Once ϕ is known, the rate of growth of capital is determined entirely within the I -sector (somewhat like Ricardo's corn rate of profit). This rate of growth is a constant given a constant equilibrium value of ϕ . But a constant rate of growth of capital implies a continuously diminishing marginal productivity of capital in the C -sector. In the face of a constant marginal productivity A of capital in the I -sector, however, the diminishing productivity in the C -sector can be sustained by competitive markets only if p , the price of capital relative to C , falls over time. This follows because a market equilibrium imposes the no arbitrage condition that capital be allocated between the two sectors so that the (value of) its marginal product be equal in the two sectors, i.e.,

$$pA = \alpha B(\phi K)^{\alpha-1} L^{1-\alpha} \quad (20)$$

³⁰The terminology "indecomposable" has the same connotation as in input-output theory.

Hence,

$$g_p = \frac{\dot{p}}{p} = (\alpha - 1)g_k, \quad (21)$$

where, according to (18), the rate of growth of capital satisfies

$$g_k = \frac{\dot{K}}{K} = A(1 - \phi).$$

Similarly, (19) implies that g_c , the rate of growth of C , satisfies

$$g_c = \alpha g_k. \quad (22)$$

Thus, $g_c < g_k$.³¹

Finally, a second no arbitrage condition ensures that the rate of interest for consumption good denominated loans (r_c) is the same as that in capital good denominated loans (r_k) corrected for the rate of change of the relative price (g_p):

$$r_c = r_k + g_p. \quad (23)$$

Replacing r_k in (23) by the marginal productivity of capital in the I sector and using in succession (21) and (22), it is possible to derive the supply rate of growth of C as a function of the rate of interest r_c :

³¹This is obvious from the relation

$$\frac{C}{K} = B(\phi)^\alpha \left(\frac{L}{K}\right)^{1-\alpha},$$

since L is a constant. While $g_c < g_k$, both are strictly positive under Rebelo's assumptions. However, for production functions reaching saturation at finite K , the positivity of g_c cannot be retained in the face of steady growth in K .

$$g_c^s(r_c) = \frac{\alpha(A - r_c)}{1 - \alpha} \quad (24)$$

Unlike the supply function in Arrow, the Rebelo function is not perfectly inelastic and it is now possible to say that both demand and supply have a role to play in determining the equilibrium rate of growth g_c^* .

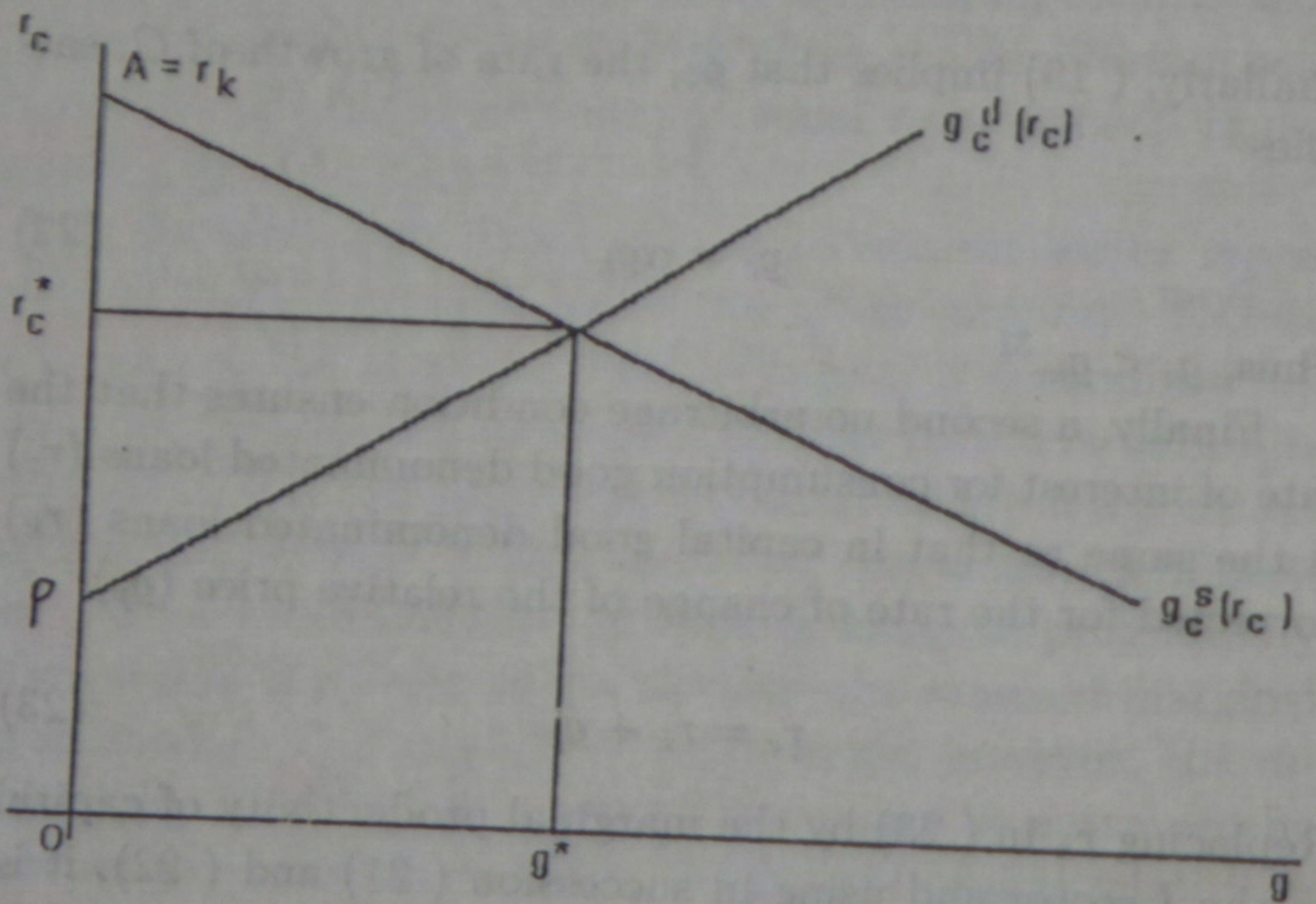


Figure 2: REBELO

Once the latter is known, (22) determines g_k^* also. Thus,

$$g_c^* = \frac{\alpha(A - \rho)}{1 - \alpha(1 - \sigma)}$$

$$g_k^* = \frac{(A - \rho)}{1 - \alpha(1 - \sigma)}$$

The diagram confirms the condition for positive rate of growth, viz., $A > \rho$. Writing $Y = C + pI$, the equilibrium rate of growth of Y is easily seen to be $g_y^* = g_c = \alpha g_k$. The appearance of preference parameters in the expressions for g_c^* etc. indicates that savings behaviour plays an important role in determining the equilibrium rate of growth. As in the d'Autume and Michel case, a tax on income, by affecting the saving rate, will have an impact on the growth rate.³²

Rebelo's results therefore yield the following Proposition:

PROPOSITION 10 *Even in the presence of strong diminishing returns in parts of the economy, steady state capital accumulation is possible at an endogenously chosen rate so long as there is an indecomposable subsector of the economy which produces capital goods by using capital goods alone. All other sectors into which the capital good enters as an input and where production is carried out under diminishing returns, adjust to this rate of growth through a continuous fall in the relative price of capital goods.*

³²It is of some interest to note here that the Rebelo story will remain valid even when L grows at an exogenously specified rate $n > 0$. Exactly the same sequence of arguments can be used to show that

$$g_c = \frac{\alpha(A + \frac{1-\alpha}{\alpha}n - \rho)}{1 - \alpha(1 - \sigma)}$$

$$g_k = \frac{(A + \frac{1-\alpha}{\alpha}n - \rho)}{1 - \alpha(1 - \sigma)}$$

In interpreting K as a composite of human and nonhuman capital, Rebelo is in a way offering a theory of technical change. Any tendency for diminishing returns to physical capital is being reversed by the rise in productivity on account of human capital growth, giving rise to constant returns in the I sector. As a theory of technical change this is rather vague in comparison to Arrow's, or for that matter Rebelo's own extension of Lucas' model. Strictly speaking, it is the constant returns technology that is at the heart of the result and it does not seem fruitful to read any more meaning into it.

3.3 Lucas, Rebelo and Uzawa: Technical Progress as Formation of Human Capital

Lucas (1988) follows Uzawa (1965) in tracing the increase in labour efficiency to a conscious attempt on the part of the work force to improve its skill level. Output is produced by means of human capital (rather than pure labour) and physical capital. With diminishing returns, the rate of growth of capital adjusts to the rate of growth of human capital, but the latter rate of growth is now determined endogenously by the model. As before, the household sector's description remains unaltered and the demand rate of growth is given by Proposition 2.³³ Denoting the aggregate level of human capital by H , the production function is written as

$$Y = AK^\beta (uH)^{1-\beta} H^\gamma \quad (25)$$

where u represents the share of "human capital time" devoted to the production of the final good and H^γ an Arrow type

³³The rate of growth of labour being zero, $c = C$.

spillover or external effect generated by the aggregate level of human capital existing in society. A fraction $(1 - u)$ of labour time is devoted to the accumulation of human capital according to the technology

$$\dot{H} = \delta(1 - u)H, \quad (26)$$

where δ is a positive constant. This sector, which produces changes in the stock of human capital, will be referred to as the H -sector.

An important similarity between Rebelo's model of the last section and Lucas' is the existence of the indecomposable subsector H determining the change in H . However, Lucas' model has two additional features, viz., a second endogenously accumulable factor K and the external effect H^γ . As is to be expected, the external effect will give rise to a divergence between the market solution and the solution for the command economy. While an extremely lucid but rigorous mathematical derivation of the results to follow may be found in Lucas (1988), an attempt will be made here to derive them exclusively with the help of Rebelo-type economic arguments by arguing in terms of implicit prices of commodities for which explicit markets are absent.

First, (25) may be rewritten

$$Y = AK^\beta (uH H^{\frac{\gamma}{1-\beta}})^{1-\beta}, \quad (27)$$

so that steady state growth involves

$$g_k = \frac{\dot{K}}{K} = \frac{1 - \beta + \gamma}{1 - \beta} \frac{\dot{H}}{H} = \frac{1 - \beta + \gamma}{1 - \beta} g_h. \quad (28)$$

where g_k and g_h stand respectively for the rates of growth of physical and human capital. Also, (9) implies that $g_k = g_c$

in a steady state, where g_c stands for the rate of growth of C .

It will be useful to concentrate on the market economy first. Denoting the (implicit) price of H relative to Y by p , efficient allocation of H between the two sectors requires that u be chosen to equate the the value of marginal contribution of H in the two sectors. This yields

$$\begin{aligned} p\delta &= (1 - \beta)AK^\beta u^{-\beta} H^{-\beta} H^\gamma \\ &= (1 - \beta)AK^\beta u^{-\beta} H^{\gamma - \beta}. \end{aligned} \quad (29)$$

Equation (29) may be differentiated to obtain

$$\begin{aligned} g_p &= \frac{\dot{p}}{p} \\ &= \beta g_k + (\gamma - \beta)g_h \\ &= \left[\beta + \frac{(\gamma - \beta)(1 - \beta)}{1 - \beta + \gamma} \right] g_k \\ &= \left[\beta + \frac{(\gamma - \beta)(1 - \beta)}{1 - \beta + \gamma} \right] g_c \\ &= \frac{\gamma}{1 - \beta + \gamma} g_c. \end{aligned} \quad (30)$$

The parallel (23) is

$$\tau_y = \delta + \frac{\gamma}{1 - \beta + \gamma} g_c \quad (31)$$

where τ_y is the rate of interest on Y denominated loans. The nonzero change in p , i.e., $\frac{\gamma}{1 - \beta + \gamma} g_c$, can arise only in the presence of diminishing returns in the Y -sector. Under the assumption of steady growth (i.e., when $g_k = \frac{1 - \beta + \gamma}{1 - \beta} g_h$),

however, there is clearly no diminishing returns in the Lucas model.³⁴ However, the market economy is unaware of this, for even along the steady state path it fails to internalize the external effect of H^γ . As is to be expected, the effect is absent in a command economy. Thus, the supply rate of growth of C for the market economy is

$$g_c^s(r_y) = \frac{(1 - \beta + \gamma)(r_y - \delta)}{\gamma}.$$

Once again, the supply rate varies with the rate of interest and one expects demand parameters to start playing a role in determining the endogenous growth rate. Combining this with Proposition 2, the equilibrium rate of growth of C is

$$g_c^* = \frac{(1 - \beta + \gamma)(\delta - \rho)}{\sigma(1 - \beta + \gamma) - \gamma}. \quad (32)$$

Finally, using (28), it follows that

$$g_h^* = \frac{(1 - \beta)(\delta - \rho)}{\sigma(1 - \beta + \gamma) - \gamma}. \quad (33)$$

To the extent that $1 \geq u \geq 0$, equation (26) implies that the maximum rate of growth of human capital possible in the Lucas model is given by ρ .

The demand and supply curves as well as the equilibrium rates of interest and growth are depicted in Figure 3. The supply curve is drawn under the assumption of a positive externality. In case the latter is negative, the curve *could* be

³⁴In Rebelo this happened due to the presence of a nonreproducible factor.

downward sloping.

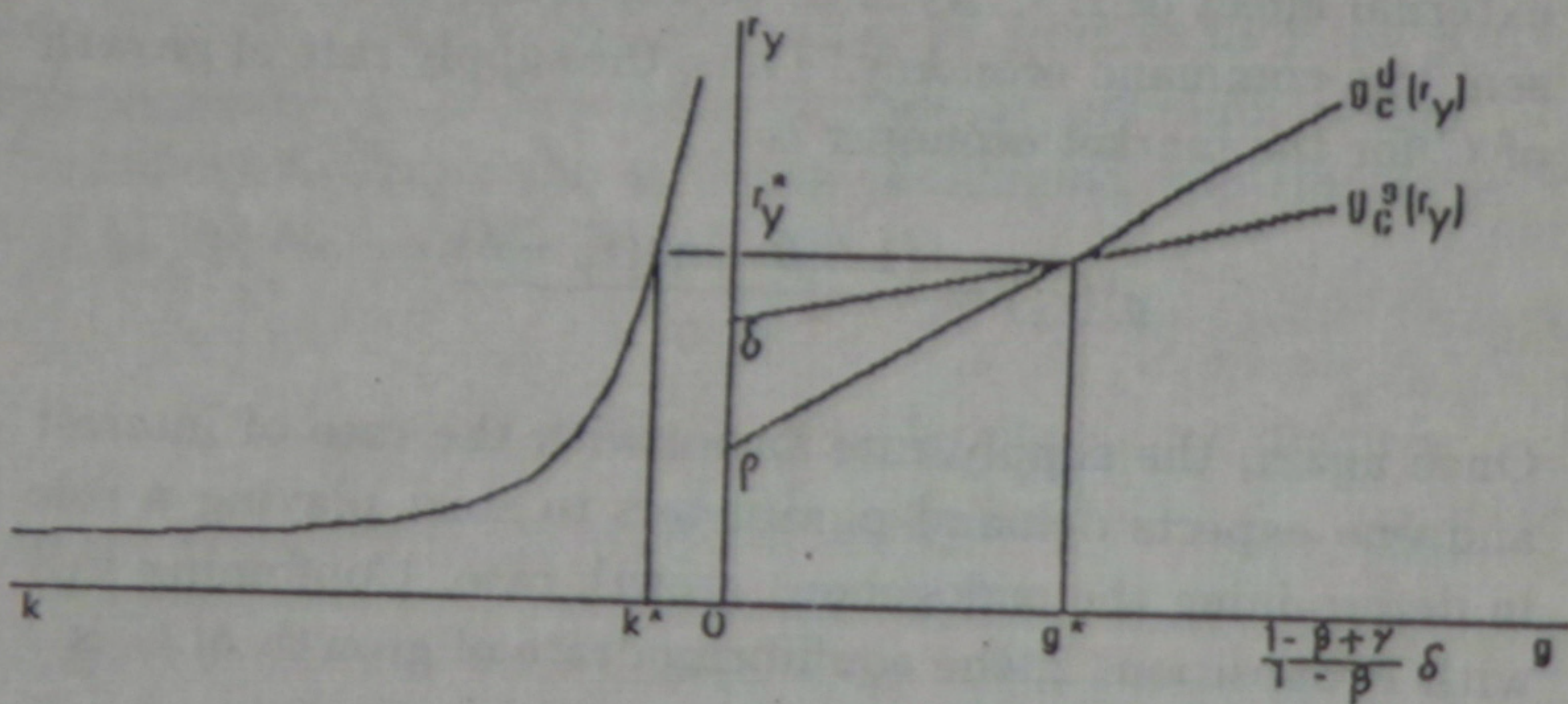


Figure 3: LUCAS- Market Solution

A set of sufficient conditions for positive solutions to exist is

(i) The slope of $g_c^s(r_y) > g_c^d(r_y)$,

i.e., $\frac{\gamma}{1-\beta+\gamma} < \sigma$; and

(ii) Intercept of $g_c^s(r_y)$ on the r_y axis $>$ intercept of $g_c^d(r_y)$,

i.e., $\delta > \rho$.

Going over to the command economy now, $\frac{\dot{p}}{p}$ is identically zero as explained earlier. A second difference arises from the planner's perception of the social marginal productivity of a unit of investment in human capital in the Y-sector. With positive externality, it is now larger than the right hand side of (29) and equals

$$\frac{\partial Y}{\partial(uH)} = (1 - \beta)AK^\beta u^{-\beta} H^\gamma \quad (34)$$

$$+ \gamma AK^\beta (uH)^{1-\beta} H^{\gamma-1} \frac{1}{u}$$

$$= (1 - \beta)AK^\beta u^{-\beta} H^{\gamma-\beta} \left(\frac{1 - \beta + \gamma}{1 - \beta} \right) \quad (35)$$

The marginal productivity of H being $\frac{1-\beta+\gamma}{1-\beta}$ times that in the market economy, the parallel of (31) for the planner is simply

$$r_y = \delta \left(\frac{1 - \beta + \gamma}{1 - \beta} \right). \quad (36)$$

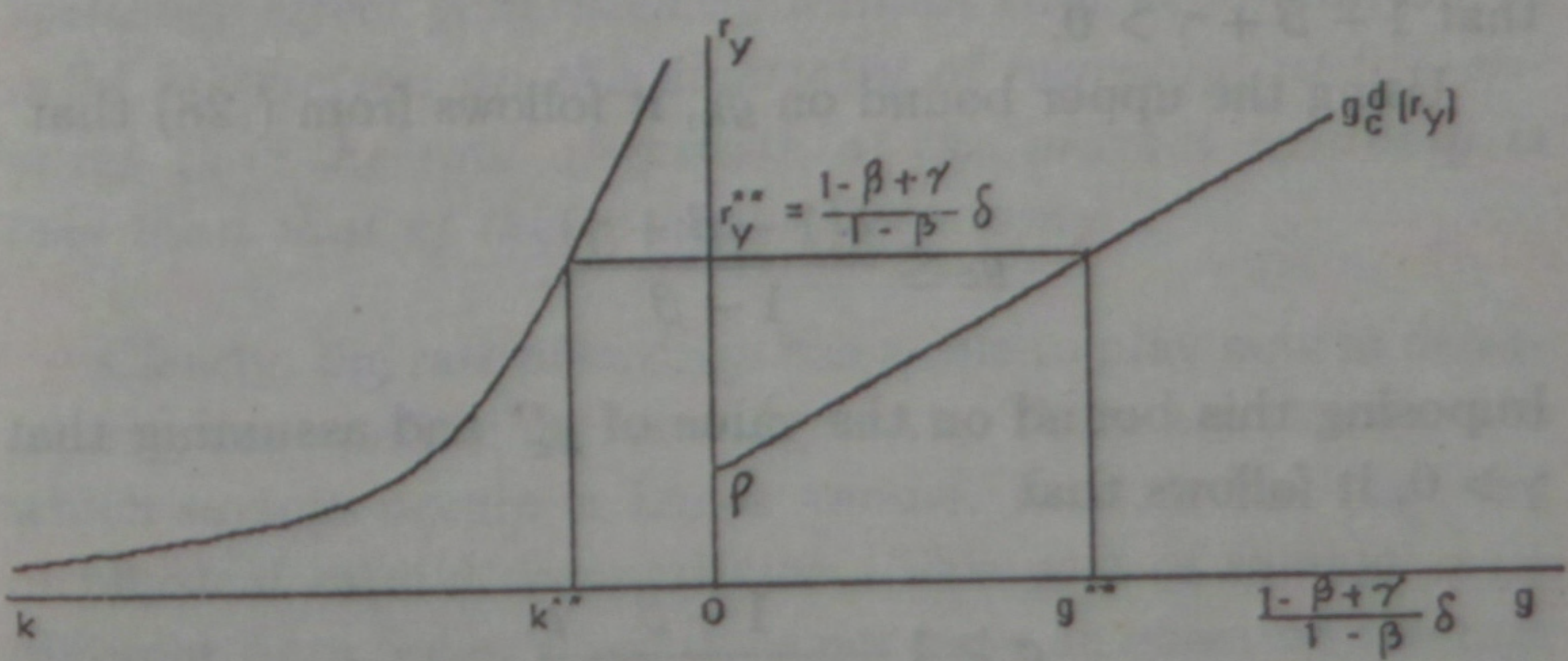


Figure 4: LUCAS- Command Solution

This implies that the supply rate of growth curve is horizontal at the level of r_y given by (36). Denoting this rate of interest

by r_y^{**} , the supply rate of growth curve is infinitely elastic at this rate. Plugging it into Proposition 2,

$$g_c^{**} = \sigma^{-1} \left(\frac{\delta(1 - \beta + \gamma)}{1 - \beta} - \rho \right) \quad (37)$$

Finally, using (28),

$$g_h^{**} = \sigma^{-1} \left(\delta - \frac{\rho(1 - \beta)}{1 - \beta + \gamma} \right). \quad (38)$$

The equilibrium is depicted in Figure 4. A sufficient condition for a positive solution to g_c^{**} to exist now is that

$$r_y^{**} = \delta \left(\frac{1 - \beta + \gamma}{1 - \beta} \right) > \rho.$$

The same condition guarantees that g^{**} is positive, provided that $1 - \beta + \gamma > 0$.

Using the upper bound on g_h , it follows from (28) that

$$g_c \leq \frac{\delta(1 - \beta + \gamma)}{1 - \beta}.$$

Imposing this bound on the value of g_c^{**} and assuming that $\gamma > 0$, it follows that

$$\sigma \geq 1 - \frac{1 - \beta}{1 - \beta + \gamma} \frac{\rho}{\delta}.$$

Finally, noting that

$$g^{**} = \frac{(1 - \beta)(\delta - \rho) + \gamma\delta}{\sigma(1 - \beta)}$$

and

$$g_c^* = \frac{(1 - \beta)(\delta - \rho) + \gamma\delta - \gamma\rho}{\sigma(1 - \beta) + (\sigma - 1)\gamma},$$

it follows that a set of sufficient conditions for $g_c^{**} > g_c^*$ is that

$$\gamma > 0 \quad \text{and} \quad \sigma > 1.$$

Lucas' essential results are summarized as

PROPOSITION 11 *In a model of endogenous human capital formation along with accumulation of physical capital, the growth rate of the system is endogenously determined by the parameters of the preference function and technology. Further, in the presence of a positive spillover effect generated by human capital formation, a mild restriction on the elasticity of marginal utility ensures that the rate of growth of the market economy is less than that of the command economy.*

Clearly, the rate of savings has a role to play now in determining the equilibrium rate of growth. There are two ways in which savings occurs in Lucas' model. First, it has the form of physical capital accumulation. This part of savings is no different from Solow's and does not have an effect on the rate of growth. But allocating labour away from the production of final goods for the sake of human capital accumulation also constitutes savings. It is this form of savings that determines the rate of growth. Rebelo (1991) builds on this model to make the technology for human capital generation dependent on physical as well as human capital. Thus, simultaneously with the allocation parameter u , he has another allocation

parameter ϕ determining the allocation of physical capital between the two sectors. This allows him to separate out the two types of savings. Also, to the extent that both physical and human capital are used in the production of human capital, both types of savings behaviour now affect the equilibrium rate of growth. It should be clear that the source of growth in this version of the Rebelo model is different from that in his linear model.

3.4 Romer: Modelling Research

As with Lucas and Rebelo, the growth rate is traced back in Romer (1990) to a core sector unencumbered by diminishing returns.³⁵ He questions, however, Lucas' conclusion that there is an upper bound on an economy's achievable rate of growth. After all, an important moving force behind economic activity is the desire to break through existing barriers on growth rates. As opposed to Lucas therefore, who locates the source of economic growth in the accumulation of human capital (by means of human capital), Romer believes that growth is not an outcome of human capital accumulation as such, but of the invention of increasingly productive techniques. Human capital is a *sine qua non* for this activity insofar as it is a vitally important input (in the shape of research labour) into the inventive process. For a given stock of human capital, there is of course an upper bound on the achievable rate of growth in Romer also, but the bound itself is flexible upwards with a growth in this stock. To emphasise this fact, Romer assumes the stock of human capital to be at

³⁵As in Arrow (Case $\alpha < 1$) and Lucas, but unlike Rebelo, the absence of diminishing returns is inseparable from the theory of technical progress offered by the model.

a constant level \bar{H} over time.³⁶

Once again, the equilibrium rate of growth is found by equating the supply rate of growth to the demand rate. There is no change in the principles underlying the latter. The supply rate too is determined, as in Rebelo, by imposing two separate no arbitrage conditions. The first equates the marginal product of human capital in the production of final goods to the value of it in research. Since the stock of human capital is a constant, however, a no arbitrage condition involving loans in human capital denominated units is no longer valid. This condition is replaced by the requirement that the rate of return on investment in research is equal to the market rate of interest.³⁷

The production function for final goods³⁸ is chosen to be

$$Y = H_y^\alpha \int_0^A x(i)^{1-\alpha} di, \quad (39)$$

where H_y represents human capital employed in producing Y , $x(i)$ the quantity of the i^{th} variety of specialized input used in the production of Y and A the cardinality of a continuum of existing varieties. Specialized inputs are indexed according to the chronological order of their appearance. The proximate

³⁶In identifying the source of technical progress in purposive research, Romer is also offering a way out of the weakness of the Arrow model, which views technical progress as an inevitable byproduct of the process of capital accumulation.

³⁷In anticipation of what follows, it is worth pointing out that in order to make research an incentive compatible activity, a monopoly element has to be introduced into the model. This would ensure that the rate of interest is no longer equal to the marginal productivity of capital.

³⁸Following an idea of Ethier (1982), the function is a reinterpretation of the Dixit-Stiglitz (1977) preference function for variety.

object of research is the i^{th} design or idea³⁹, whose concrete embodiment is the input $x(i)$. Progress in research implies an increase in the value of A and along with it, an expansion in the size of the set of ideas as well as that of the set of specialized inputs.

That (39) captures the notion of specialization may be appreciated from the following considerations. Assuming all $x(i)$'s to be used at the same level, say x , the production function reduces to

$$Y = H_y^\alpha A x^{1-\alpha}. \quad (40)$$

Given x , the marginal return to variety is measured by

$$\frac{\partial Y}{\partial A} = H_y^\alpha x^{1-\alpha}.$$

In other words, there are constant returns to variety, given the level of $x(i)$. As opposed to this, the marginal return to an increase in x , given that variety is fixed at A , behaves as follows:

$$\begin{aligned} \frac{\partial Y}{\partial x} &= H_y^\alpha A(1 - \alpha)x^{-\alpha} > 0 \\ \frac{\partial^2 Y}{\partial x^2} &= H_y^\alpha A(-\alpha)(1 - \alpha)x^{-\alpha-1} < 0. \end{aligned}$$

Thus, given A , there is decreasing returns to a mere increase in the quantity in which the different inputs are employed. In other words, specialization offsets the tendency for diminishing returns associated with an intensive use of existing inputs.

³⁹The two words "design" and "idea" will be used synonymously in what follows.

The design underlying a specialized input is nonexcludable, since it can be copied relatively costlessly.⁴⁰ This acts as a disincentive for research as a privately organized activity. Romer's model resolves the difficulty by adopting a monopolistically competitive market structure backed by patent laws. There are two ways in which an existing design may be utilised. First, it may be used to produce the specialized input based on it. Second, any idea can be utilized to create newer ideas through further research. The patent laws in Romer inhibit free access to a design by a non-patent holder in the first of these activities. However, the laws do not preclude the free use of an existing idea to churn out new ones. In other words, a patent accords to the product of invention the status of a *partially* excludable commodity. The total body of knowledge incorporated in the stock of existing ideas (of size A) is a free input into new research, but the use of each individual design is excludable in producing the corresponding input.

Romer's economy consists of three parts: a perfectly competitive sector producing Y ; for each i , a monopolist producing $x(i)$; and a competitive sector producing research ideas or designs. Before proceeding further, it is of some help to present the social accounts of the Romer model in a schematic form:

⁴⁰It is also nonrival, since two production processes based on the *same* idea can be simultaneously operated. Unlike rival inputs, there is no sense in which the two processes can be viewed as employing two *pieces* of the same idea. Being both nonrival and non-excludable, a design has the required characteristics of a pure public good.

Social Accounts for the Romer Model			
Sector	Output	Input	Market Structure
Final Goods	Y	$H_y, x(i), i \in (0, A]$	Competitive
Intermediate Goods	$x(i), i \in (0, A]$	K, i^{th} design, $i \in (0, A]$	Monopoly
Research	A	H_A, A	Competitive

Since Y is produced under perfect competition, the wages w_H^y of human capital in this sector equals its marginal product:

$$w_H^y = \alpha H_y^{\alpha-1} \int_0^A x(i)^{1-\alpha} di \quad (41)$$

Similarly, the inverse derived demand function for the input $x(i)$ is

$$p(i) = (1 - \alpha) H_y^\alpha x(i)^{-\alpha} \quad (42)$$

where $p(i)$ is the price paid per unit use of $x(i)$.

The first no arbitrage condition requires that the right hand side of (41) be equal to the value of the marginal product of human capital in research. To calculate the latter, the details of the research technology have to be specified along with the market structure. The production of $x(i)$ may be thought of as a two stage (but not necessarily vertically integrated) process. The lower stage consists of the creation of a design, i.e., the task of "hitting an idea". Human capital is an essential input in this process and calls forth a once for all payment to the inventor. At the higher stage, nonspecialized or raw capital (Solow's K) is converted to specialized $x(i)$ by means of a fixed coefficient production function, each unit of $x(i)$ requiring ζ units of raw capital.

The raw capital congealed in an $x(i)$ is borrowed from the household sector. If an $x(i)$ is infinitely lived, there is a variable interest cost per period to be incurred by its producer.

for infinite time. Alternatively, $x(i)$ could also be looked upon as a consumable input, reproduced in each period. In this case too, there is an infinite stream of interest payments.⁴¹ A patent holder of the i^{th} idea is the monopolist supplier of $x(i)$ who earns an infinite stream of profits. The discounted present value of this infinite stream is set against the once for all payment for the design to define the rate of return on investment in research. Potential monopolists compete for this return as soon as a new design makes its appearance. This raises the price of the design, driving the rate of return to equality with the market rate of interest.

The per period profit of a monopoly producer of $x(i)$ is given by

$$\pi(i) = p(i)x(i) - r\zeta x(i), \quad (43)$$

where r is the (real) market rate of interest paid on the raw capital $\zeta x(i)$ borrowed from the households. Using (42), the maximization of (43) for each i yields an expression for $p(i)$ that depends only on r for each i :

$$p = p(i) = \frac{r\zeta}{1 - \alpha}. \quad (44)$$

By the same logic, this is true for the value of $x(i)$ too. It depends only on r . This may be indicated by $x(r)$. The value of $\pi(i)$ is also dependent on r alone:

$$\pi = px - r\zeta x = \frac{r\zeta}{1 - \alpha}x - r\zeta x = \alpha px. \quad (45)$$

⁴¹The input $x(i)$ does not undergo obsolescence in Romer. Given the nature of the production function, there will always be positive demand for it by the competitive producers of Y . Young (1993) describes this aspect in terms of complementarity of inputs. Obsolescence related problems are recognized by Aghion and Howitt (1992) and Young.

and the infinite stream of π , discounted at the market rate of interest, equals

$$\begin{aligned} \int_0^{\infty} e^{-rt} \pi dt &= \frac{\pi}{r} \\ &= \frac{\alpha p x}{r} \\ &= \frac{\alpha(1-\alpha)H_y^\alpha x^{1-\alpha}}{r} \end{aligned} \quad (46)$$

The technology for research is given by

$$\dot{A} = \delta A H_A \quad (47)$$

where δ is a positive constant and H_A the human capital employed in producing new ideas (i.e., \dot{A}). As already indicated, this sector is perfectly competitive. Since the production function is homogeneous of degree 2 in A and H_A , payment of factors according to their marginal productivity will over-exhaust the output value. This is avoided however, A being a free input into the research process by the assumption of partial excludability. There is thus no difficulty in paying H_A according to its marginal productivity. Denoting the price of \dot{A} by P_A , the value of the marginal product of H_A is $P_A \delta A$. In order to apply the first no arbitrage condition then, it only remains to calculate P_A using the second no arbitrage condition. Since P_A is none other than the investment in research, the second no arbitrage condition implies that in equilibrium it equals the infinite stream of monopoly profits discounted at the *market rate of interest*, viz., (46).

Thus,

$$w_H^A = \delta A \alpha (1 - \alpha) H_y^\alpha \frac{x^{1-\alpha}}{r} \quad (48)$$

where w_H^A is the wages received by H_A . By the no arbitrage requirement, $w_H^y = w_H^A$, imposing which

$$H_y = \frac{r}{\delta(1-\alpha)}. \quad (49)$$

Substituting (49) into (47),

$$\frac{\dot{A}}{A} = \delta\left(\bar{H} - \frac{r}{\delta(1-\alpha)}\right). \quad (50)$$

The last step in the derivation of the supply rate function is to identify it with (50). Going back to (39), at any fixed value of r ,

$$\begin{aligned} Y &= H_y^\alpha A x(r)^{1-\alpha} \\ &= H_y^\alpha A^\alpha A^{1-\alpha} x(r)^{1-\alpha} \\ &= H_y^\alpha A^\alpha (Ax(r))^{1-\alpha} \\ &= \frac{1}{\zeta^{1-\alpha}} H_y^\alpha A^\alpha K^{1-\alpha}, \end{aligned} \quad (51)$$

where, $K = \zeta Ax(r)$ is the aggregate value of raw capital embodied in the specialized inputs when the rate of interest is fixed at r . Clearly, $\frac{A}{K}$ is fixed, given r . Using (51), the counterpart of (9) in Romer's model is

$$\frac{C}{K} + \frac{\dot{K}}{K} = H_y^\alpha \left(\frac{A}{K}\right)^\alpha, \quad (52)$$

which means that at any given r , the rates of growth of A , C , K and Y are the same. Thus, (50) is equivalently rewritten as

$$g^s(r) = \delta\left(\bar{H} - \frac{r}{\delta(1-\alpha)}\right). \quad (53)$$

As with Rebelo, this is a declining function of r .

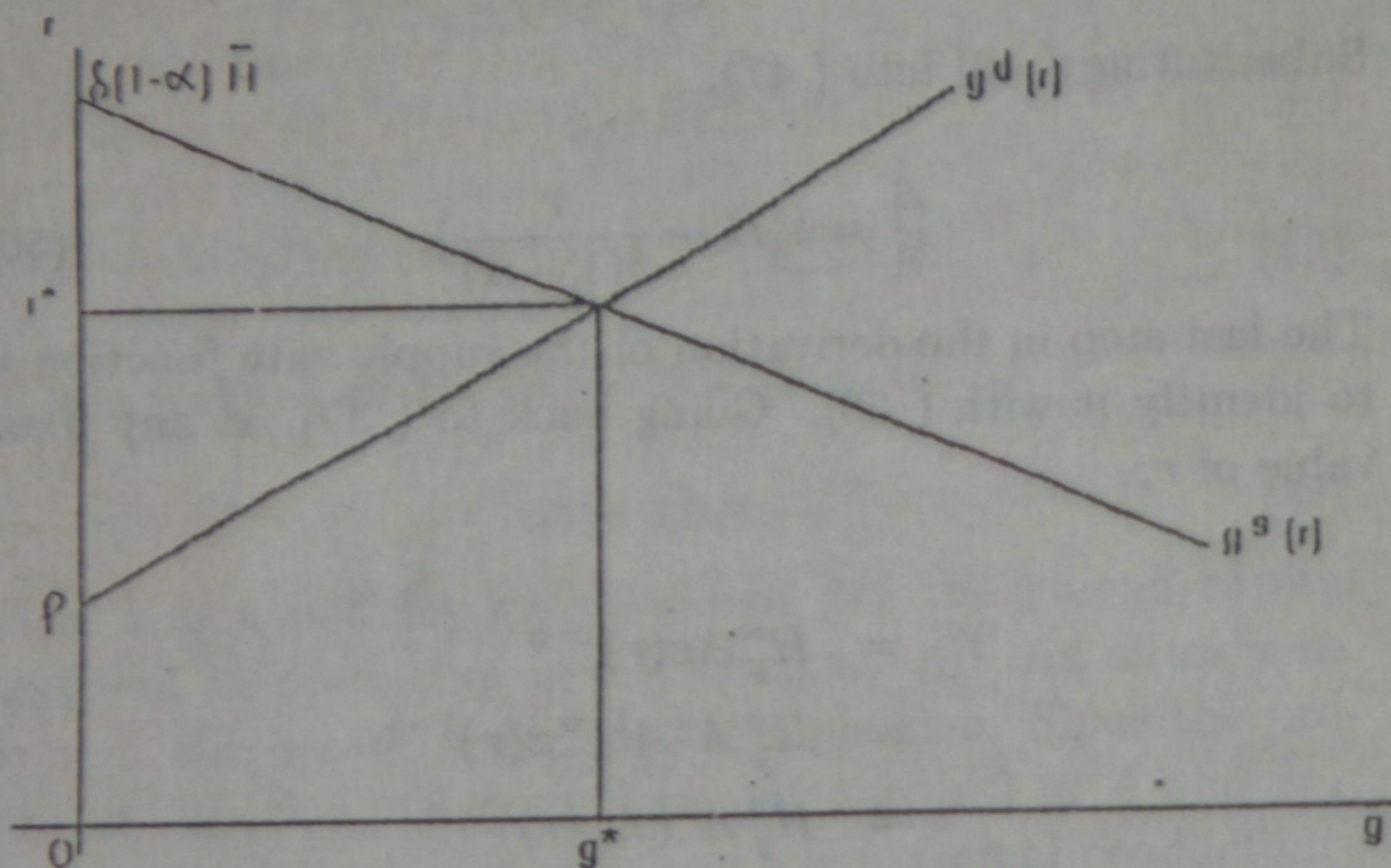


Figure 5: ROMER (1990)

The equilibrium is shown in Figure 5. The diagram shows that a positive rate of growth is achievable in equilibrium only if H is large enough.⁴² Romer argues on the basis of this observation that a minimum size of the stock of human capital is a necessary precondition for growth. The difference between rich nations and the poor may be explainable therefore in terms of the differences in the size of human capital in these societies.

⁴²To be precise, the condition is $\delta^2(1 - \alpha)\bar{H} > \rho$.

Finally, there would be a divergence between the social and private marginal products of x due to the facts that (i) the market economy will not internalize the intertemporal externality generated by the investment in x (an investment today raises productivity for all time to come) and (ii) a monopolistic equilibrium leads to a socially suboptimal allocation of resources.

Romer's findings may now be summarized.

PROPOSITION 12 *When technical progress assumes the form of specialized inputs into the productive process, the equilibrium growth rate of an economy can be sustained by a combination of perfectly competitive and monopolistically competitive markets, the latter backed by a legal structure recognizing patent rights. The rate of growth for the command economy is larger than that for a market economy.*

3.5 Aghion and Howitt: Obsolescence

Apart from its being in the nature of a public good, a further disincentive to research is the fact of obsolescence. Newer and more efficient techniques replace the old with the result that efforts devoted towards invention of better methods of production are negatively affected by the possibility of even better methods appearing in future. So long as a newly developed technique holds sway, however, monopoly profits can be enjoyed from its use. While Romer assumes that any specialized input, once invented, is used forever, Aghion and Howitt (1992) go to the other extreme and assume that a specialized input goes into extinction as soon as a newer variety appears. The cause behind replacement is the higher productivity of

new inputs.⁴³

The model, however, specifies the rate of increase in productivity exogenously. As such, the rate of growth of productivity is not even a variable of the model. What is not known with certainty, however, is the arrival time of a new input. Thus, even though the growth potential of a new input is known exactly, it is possible to speak only in terms of an *expected* growth rate of the economy. The Aghion and Howitt exercise is geared towards a determination of the equilibrium value of this expected rate of growth. The model is entirely supply oriented, the demand rate of growth having no role to play even in a trivial sense (as in Solow above, for example). Thus the model is not a full equilibrium model and the real rate of interest r is exogenously given.

Time is denoted by the continuous variable τ and inventions by the discrete variable t . Thus, x_t denotes the specialized input relevant for time *interval* t , x_{t+1} for *interval* $t+1$. The length of any such interval is a stochastic variable.

As in Romer, there are three sectors of production, the final good sector, the intermediate good sector and the research sector. The final good is produced according to

$$Y = AF(x), \quad (54)$$

where A is a productivity parameter whose level is determined by the type of x in use.⁴⁴ Thus, x_t is associated with A_t , x_{t+1} with A_{t+1} , and so on. Further, $A_{t+1} = \gamma A_t$, $\gamma > 1$, where γ is a nonstochastic variable. If successive generations

⁴³They also consider the possibility of simultaneous use of different varieties. In this context, see Young (1993) also. In Romer, the productivity differential is absent.

⁴⁴Production of Y requires unskilled labour also, but this is a constant and hence ignored.

of the inputs were to be available in a deterministic chronological sequence, the economy's growth rate would be fairly transparent. However, as noted already, this is not the case, the arrival time of a new input being a stochastic variable.

The entire quantum of Y is consumed so that there is no direct savings out of Y . Nevertheless, savings enters through the back door since the level of Y depends on that of x and the latter is determined by the allocation of skilled labour to the production of x . The larger (smaller) the allocation in favour of x , the larger (smaller) is current consumption. Thus, a decision to produce less of x amounts a decision to save more.

The input x is produced by means of skilled labour but the latter has a competitive use in research also. A fixed quantity N of skilled labour is allocated between these alternative uses. (In Romer, it is allocated between research and the final good).

The two activities will be described in turn and it is simpler to begin with x . Any x is produced by the linear technology:

$$x = L. \quad (55)$$

As with Romer, each x_t producer is a monopolist, so that the equilibrium condition for this sector is given by the equality of marginal cost (i.e., the wage rate w_t expressed in Y units) and revenue, viz.,

$$w_t = A_t[F'(x_t) + x_t F''(x_t)], \quad (56)$$

or,

$$\omega_t = [F'(x_t) + x_t F''(x_t)] = \tilde{\omega}(x_t), \quad (57)$$

where ω_t stands for the productivity adjusted wage rate during t . Combining with (56), the last equation can be written

$$w_t = A_t \tilde{\omega}(N - n_t). \quad (58)$$

By assumption, only one specialized input is in use at any time. At each point of time, the equilibrium profit accruing to the monopolist producing x_t is written π_t , while $\tilde{\pi}(\omega_t)$ [= $\tilde{\pi}(\tilde{\omega}(x_t))$] represents the optimal value of $\frac{\pi_t}{A_t}$ as a function of ω_t . As w_t increases, the profit maximising value of x_t falls, thus reducing the quantum of labour used in this sector. In Figure 4, the upward rising curve in the right hand panel shows the relationship (58) between w_t and the amount of labour available for research, given that the optimum amount of x_t is being produced. The intercept of the curve on the vertical axis is given by $w_t = A_t \tilde{\omega}(N)$. At any wage rate below this, the entire N is devoted to producing x_t . For later use, it should be noted here that $\tilde{\pi}(\omega_t)$ is a (monotone decreasing) function of ω_t .

This completes the description of the x sector. Going over now to research, the interesting innovation of the paper lies in treating the arrival time of new ideas, i.e., the output of research, as a random variable. The probability of success (in research) at any point of time is assumed to have a Poisson distribution with parameter $\lambda\phi(n)$, where ϕ is a linear function of n ($\phi' > 0$), the labour devoted to research at each point of time. There is perfect competition in the research sector and V_{t+1} is the price of the $t + 1^{th}$ innovation, i.e., the *idea* behind the specialized input x_{t+1} . A firm engaged in producing this idea faces a probability $\lambda\phi$ of success at any τ , so that $1 - \lambda\phi$ is the probability that the innovation does not occur. In case of success, the revenue

earned is V_{t+1} , otherwise the revenue is zero. In either case, however, it has to incur a wage cost equal to $w_t n_t$.⁴⁵ Thus, $\lambda\phi(n_t)V_{t+1} + (1 - \lambda\phi(n_t)) \cdot 0 = V_{t+1}\lambda\phi(n_t)$ stands for the *expected* revenue and $w_t n_t$ for the *deterministic* cost at each τ in t . Hence, the objective function of a firm engaged in research is given by:

$$V_{t+1}\lambda\phi(n_t) - w_t n_t. \quad (59)$$

As before, there are two no arbitrage conditions relevant for the determination of the (expected) supply rate of growth. First, at each τ in t , the wage rate w_t in the monopoly sector must equal the wage rate in the research sector. The second no arbitrage condition states that the investment V_{t+1} made by a potential monopolist in buying the output of research must yield a rate of return equal to the market rate of interest r .

The research sector being competitive, the wage rate of research labour equals the *expected* value of its marginal product.⁴⁶ Thus, the first of the no arbitrage conditions yields

$$w_t = V_{t+1}\lambda\phi'(n_t). \quad (60)$$

In order to formalise the second no arbitrage condition, the stream of expected returns associated with an investment V_{t+1} has to be identified. This depends on the expected length

⁴⁵It is worth emphasising that n_t stands for labor engaged in producing the $t + 1^{\text{th}}$ innovation.

⁴⁶The wage rate in this sector has to be interpreted in the sense of an *imputed* value, since the *actual* payment is held up till the research bears fruit. A capital market structure that sustains the process is described later.

of the interval $t + 1$ for which x_{t+1} yields returns and this in turn depends on n_{t+1} , the labour engaged in inventing x_{t+2} . Thus, the expected return at any τ in $t + 1$ is $e^{-\lambda\phi(n_{t+1})\tau} \pi_{t+1} = e^{-\lambda\phi(n_{t+1})\tau} \pi(\bar{\omega}(N - n_{t+1}))$, since π_{t+1} must correspond to the optimum value of profit at τ . Consequently, the discounted present value of this expected stream is

$$\begin{aligned} V_{t+1} &= \int_0^{\infty} e^{-r\tau} e^{-\lambda\phi(n_{t+1})\tau} A_{t+1} \bar{\pi}(\bar{\omega}(N - n_{t+1})) d\tau \\ &= \frac{A_{t+1} \bar{\pi}(\bar{\omega}(N - n_{t+1}))}{r + \lambda\phi(n_{t+1})} \end{aligned} \quad (61)$$

Putting (60) and (61) together,

$$w_t = \lambda\phi'(n_t) \frac{A_{t+1} \bar{\pi}(\bar{\omega}(N - n_{t+1}))}{r + \lambda\phi(n_{t+1})} \quad (62)$$

Definition of Equilibrium

An equilibrium solution path for research is a sequence $\{n_t\}$, $t = 1, 2, 3, \dots$ such that each successive pair $\{n_t, n_{t+1}\}$ satisfies (62).

For every (expected) value of n_{t+1} , (62) gives the value of w_t required to satisfy the first no arbitrage condition. Plugging this into (56), one gets n_t , the allocation of labour into research at each τ during t .⁴⁷ Since, $\bar{\pi}$ is monotone decreasing in ω , a rise in n_{t+1} reduces w_t and hence n_t . The relation

⁴⁷The maximum may not be interior, in which case (62) does not hold as an equality.

between n_t and n_{t+1} is captured by Figures 6 and 7.

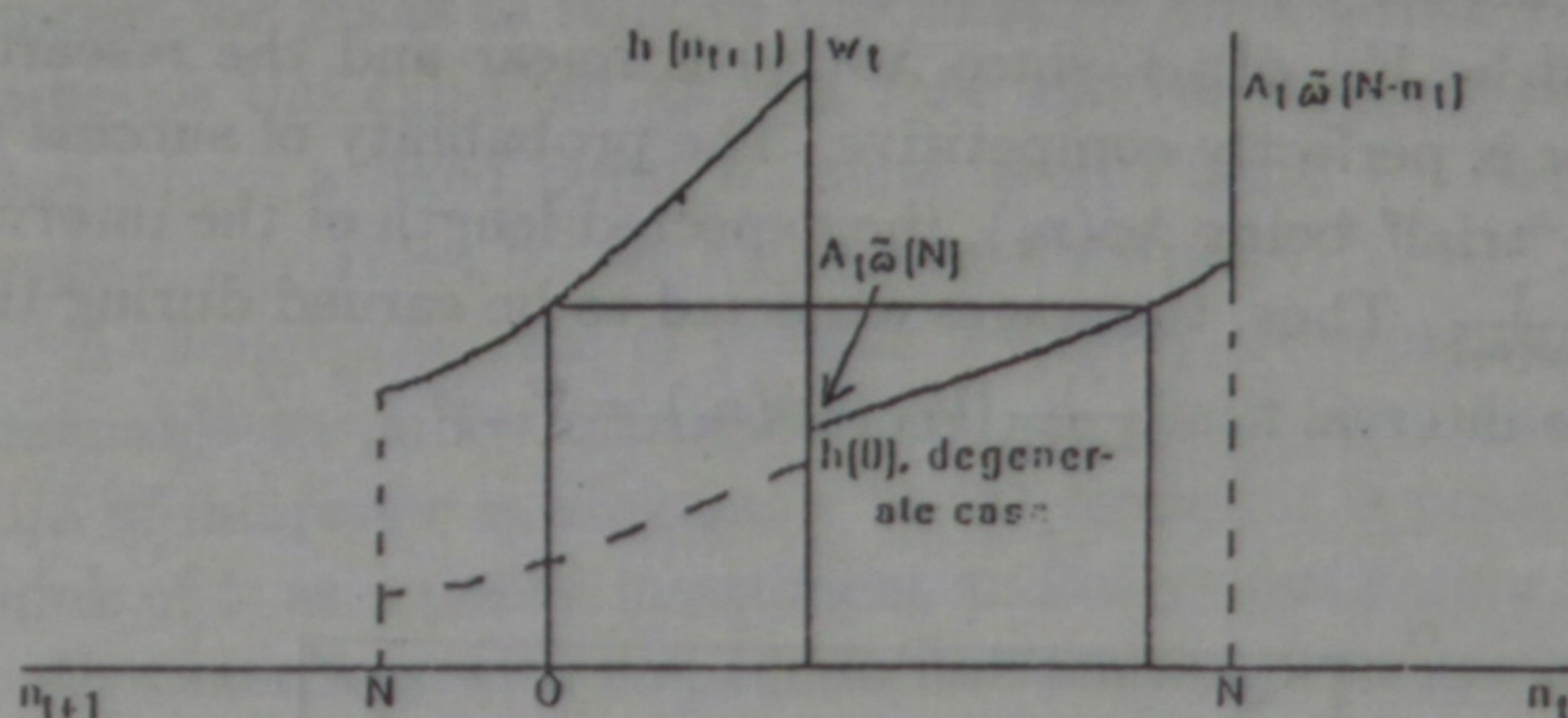


Figure 6: AGHION-HOWITT

Figure 5 gives a fixed point for the problem, with the same level of equilibrium n in all periods.

Denoting the right hand side of (62) by $h(n_{t+1})$, a possibility of zero research is shown by the broken curve in the left hand panel of Figure 6. This happens only when $h(0) < A_t \tilde{\omega}(N)$.

An interesting feature of the model arises from the fact that research produces a revenue only when its output, say, the $t + 1^{\text{th}}$ innovation, actually arrives. Prior to that point of time, research labour cannot be paid for from the value of its produce. The problem may be resolved by assuming that the monopoly firm producing x_{t+1} is owned by the labourers producing the $t + 1^{\text{th}}$ idea. They are paid in terms of shares in the profits of their firm once the innovation is successful. The discounted present value of their expected earning

would then be V_{t+1} . This is exactly the same as their expected wage earnings during the interval t as labourers in the research firm. This may be seen as follows. Had they been actually paid their wages, the earnings at each τ in t would be $V_{t+1}\phi(n_t)$, since $\lambda\phi(n_t)$ is linear and the research sector is perfectly competitive. The probability of success in each "trial" being $\lambda\phi(n_t)$, the expected length of the interval t is $\frac{1}{\lambda\phi(n_t)}$. Thus, the wages expected to be earned during the entire interval t is $[\frac{1}{\lambda\phi(n_t)}]V_{t+1}\lambda\phi(n_t) = V_{t+1}$.

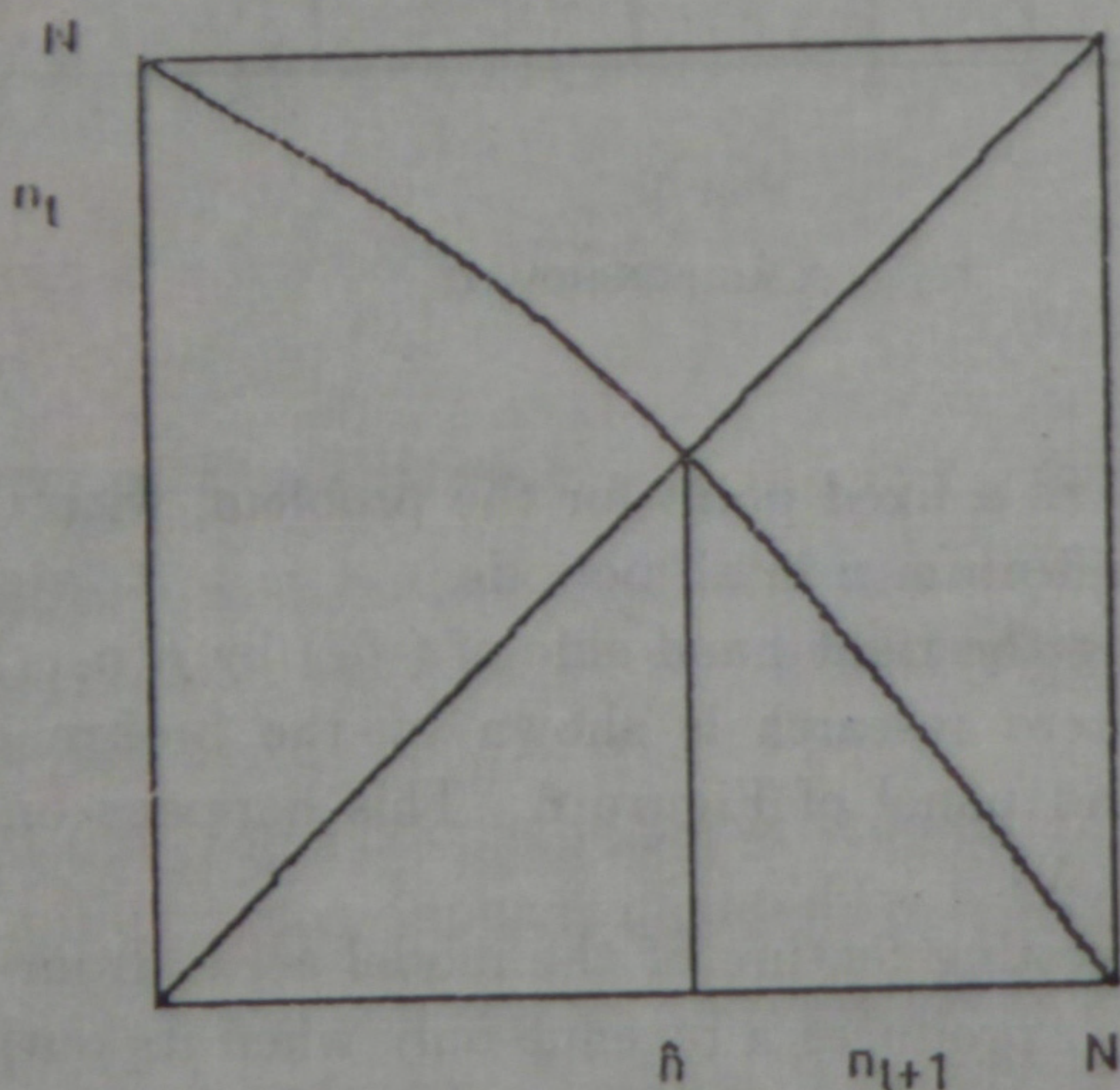


Figure 7: AGHION-HOWITT

The question nevertheless remains as to how the research

labour n_t is actually sustained during t , since the wages cannot be paid. One must assume then that there is a source of finance for research labour which is replenished later as soon as the innovation is successful. One can also try and add a market for loans to remove the lacuna. Aghion and Howitt compute the expected growth rate and its variance for the case where a fixed point \hat{n} exists as in Figure 7. They are respectively given by $\lambda\phi(\hat{n})\ln\gamma$ and $\lambda\phi(\hat{n})(\ln\gamma)^2$.

Although the rate of growth is supply determined, it is amenable to policy manipulations. In particular, since the rate of interest r is exogenous to the model, it is possible to think of it as a policy instrument and carry out comparative statics exercises with respect to its variations. In particular, following the arguments developed, it is not hard to see that the fixed point \hat{n} must fall with a rise in the rate of interest. With a rise in r , the value of marginal product of research labour falls. This causes the wage rate consistent with equilibrium allocation to fall and hence the profit maximising employment in the x sector to rise. Consequently, research employment falls.

As opposed to the previous models, a social optimum may involve a *lower* rate of growth than the market equilibrium. This happens because in a market economy, researchers do not internalize the negative effect of inventions on existing profits.

Thus, the major results of Aghion and Howitt may be summed up as follows:

PROPOSITION 13 *In the presence of obsolescence, the arrival of more productive inputs has a depressing effect on the rate of private investment into research. The growth rate of the economy is a stochastic variable and the ex-*

pected rate of growth of a market economy might exceed that of the command economy, since the latter internalizes the destructive effect of the arrival of new ideas on existing productive processes.

3.6 Barro: Public Sector and the Role of Infrastructure

It has often been claimed that inadequate infrastructure constitutes a serious constraint on growth. Barro (1990) can be viewed as an attempt to capture this problem. The resulting model leads back to a demand driven growth rate, the supply rate playing no role at all.

The two arguments in the production function are now capital and a public sector input. The latter is in the nature of a public consumption item of national accounts, entering both as a factor payment on the cost side and a value added item on the revenue side. Denoting this by G , the production function is

$$Y = F(K, G). \quad (63)$$

It has all the properties of the Solow production function.

The public sector input is financed by a proportional income tax τ , i.e.,

$$\tau Y = G. \quad (64)$$

Since (63) reduces to (8), where $y = \frac{Y}{G}$ and $k = \frac{K}{G}$, equation (64) fixes k once τ is given. This in turn fixes $f'(k)$ and hence the rate of growth via Proposition 2.

The problem then reduces to a choice of the optimal τ and Barro goes on to compare the welfare maximizing τ with

the growth rate maximizing τ and shows that in general, the two may well be unequal.

A divergence arises between the social and the market equilibrium once again since in calculating the marginal product of capital, the market fails to take note of the proportionate relationship between K and G on account of a given τ . As a result, the command economy will grow at a higher rate than the market economy.

The essential features of Barro's conclusions are:

PROPOSITION 14 *When the government taxes private sector income and transfers the proceeds entirely in the form of infrastructural support, the endogenous rate of growth of the economy is a function of the tax rate. The growth maximizing tax rate may fall short of the welfare maximizing one. The command economy chooses a higher growth rate than the market economy for any given tax rate.*

4 Conclusion

As an economist from third world, I am naturally interested in models of growth and development. The important question to me therefore is whether some of the models covered here have a bearing on the everyday problems which concern us. Or at least, whether these models might be made to focus on a few of these issues. A belief shared by many policy planners saddled with a low level growth trap is that the panacea lies in an increase in the savings rate. Such a viewpoint suggests that there is no immediate prospect of capital saturation, as in the core sector of Rebelo's model. However, for this to be true, it must also be the case that the process

of capital accumulation can itself reverse all tendencies for diminishing returns. If so, the capital in question cannot be a simple one-to-one transformation of savings as in Solow. Rebelo's explanation lies in identifying a part of it as human capital. A message from NGT that is relevant for developing economies then, is that a mere effort to raise the saving rate is not going to be of help. One must at the same time be able to transform savings into increasingly sophisticated capital "equipment".

On the negative side, a crucial policy problem facing a society such as ours is that of unemployment. A perceptive reader would have noticed that the entire literature on NGT is based on the assumption of full employment. Given its concern for steady states, this is perhaps a natural assumption to make. But the nagging question then is whether an excessive concern with maintainable long term growth rates is the real issue for developing economies. In my opinion, a model of endogenous growth with unemployment would be a welcome addition to the existing literature.

There are other issues of contemporary interest. For example, developed economies are becoming increasingly conscious of the trade-off between growth and environmental quality. Is growth necessarily a desirable objective? The question that arises here is whether a developing economy can afford the luxury of a low growth rate in pursuit of a better environment. Going by the example of India, higher growth rates are taken to be indices of successful government policy. It seems unlikely that a government with a concern for environment at the cost of growth would have the sympathy of the electorate. There is an increasing body of literature on growth and environment that is of interest in this connection. It could not be covered here given the limited purpose of the

paper.

The paper is dedicated to a great teacher. It is best viewed, therefore, as an aid to teaching. It contains no new results at all, but it offers a simple way of consolidating existing results. One hopes that the scheme would be of help to those who wish to get acquainted with the essential ideas before plunging into a detailed study of the subject.

5 References

- Aghion, P. and P. Howitt (1992) A Model of Growth through Creative Destruction, *Econometrica*, pp. 323-51.
- Arrow, K.J. (1962) The Economic Implications of Learning by Doing, *Review of Economic Studies*, pp. 155-73.
- Barro, R (1990) Government Spending in a Simple Model of Endogenous Growth, *Journal of Political Economy*, pp. S103-S125.
- Barro, R and X. Sala-i-Martin (1995) *Economic Growth*, McGraw-Hill, New York.
- Becker, G. and R. Barro (1988) A Reformulation of the Economic Theory of Fertility, *Quarterly Journal of Economics*, pp. 1-25.
- Cass, D. (1965) Optimum Growth in an Aggregative Model of Capital Accumulation, *Review of Economic Studies*, pp. 233-40.
- d'Autume, A and P. Michel Endogenous Growth in Arrow's Learning by Doing Model, *European Economic Review*, 1993, pp. 1175-84.
- Denison, Edward F. (1974) *Accounting for United States Economic Growth, 1929-1969*, Washington D.C., The Brookings Institution.

Dixit, Avinash K. and Joseph E. Stiglitz (1977) Monopolistic Competition and Optimum Product Diversity, pp. 297-308.

Domar, Evsey D. (1946) Capital Expansion, Rate of Growth and Employment, *Econometrica*, 14, pp. 137-47.

Drandakis, E.M. and E.S. Phelps (1966) A Model of Induced Investment, Growth and Distribution, *Quarterly Journal of Economics*, pp. 823-39.

Grossman, Gene M. and Elhanan Helpman (1991) Innovation and Growth in the Global Economy, MIT Press, Cambridge.

Ethier, Wilfred J. (1982) National and International Returns to Scale in the Modern Theory of International Trade, *American Economic Review*, pp. 389-405.

Harrod, Roy (1939) An Essay in Dynamic Theory, *Economic Journal*, pp. 14-33.

Keynes, John M. (1936) *The General Theory of Employment, Interest and Money*, Harcourt, Brace and World, Inc., 1965.

Koopmans, T.C. (1965) On the Concept of Optimal Economic Growth, in *Study Week on the Econometric Approach to Planning, Pontificiae Scientiarum Scripta Varii XXVIII*, Rand McNally, Chicago.

Lucas, R.E. Jr. (1988) On the Mechanism of Economic Development, *Journal of Monetary Economics*, pp. 3-42.

Phelps, E.S. (1961) The Golden Rule of Accumulation: A Fable for Growthmen, *American Economic Review*, pp. 638-43.

Phelps, E.S. (1962) The New View of Investment: A Neoclassical Analysis, *Quarterly Journal of Economics*, pp. 548-67.

Phelps, E.S. (1965) Second Essay on the Golden Rule of

- Accumulation, *American Economic Review*, pp. 793-814.
- Ramsey, Frank P. (1928) A Mathematical Theory of Saving, *Economic Journal*, pp. 543-59.
- Rebelo, S. (1991) Long-Run Policy Analysis and Long-Run Growth, *Journal of Political Economy*, pp. 501-21.
- Romer, P.M. (1986) Increasing Returns and Long-Run Growth, *Journal of Political Economy*, pp. 1002-37.
- (1990) Endogenous Technical Change, *Journal of Political Economy*, pp. S 71-S102.
- Shell, Karl (ed.): *Essays on the Theory of Optimal Economic Growth*, MIT, 1967.
- Sheshinski, Eytan Optimal Accumulation with Learning by Doing, in Shell (1967), pp. 31-52.
- Solow, Robert (1956) A Contribution to the Theory of Economic Growth, *Quarterly Journal of Economics*, pp. 65-94.
- (1957) Technical Change and the Aggregate Production Function, *Review of Economics and Statistics*, pp. 312-20.
- (1960) Investment and Technical Progress, in K.J. Arrow, S. Karlin and P. Suppes (ed.) *Mathematical Methods in the Social Sciences*, Stanford, pp.89-104.
- Uzawa, H (1965) Optimum Technical Change in an Aggregative Model of Economic Growth, *International Economic Review*, pp. 19-31.
- Young, A (1993) Substitution and Complementarity in Endogenous Innovation, *Quarterly Journal of Economics*, pp.775-807.

6 Appendix

Proof of Proposition 2.

The path of optimal consumption and savings is obtained by maximizing (4) subject to (5). The problem is solved by maximizing the current value Hamiltonian

$$H(K(t), c(t), \eta(t)) = L(t) \frac{1}{1-\sigma} (c(t)^{1-\sigma} - 1) + \eta(t) [r(t)K(t) + w(t)L(t) - L(t)c(t)] \quad (\text{A. 1})$$

at each t with respect to $c(t)$. Dropping the t 's for notational convenience, this yields⁴⁸

$$c^{-\sigma} = \eta. \quad (\text{A. 2})$$

At the same time, the costate variable η is required to satisfy the condition

$$\dot{\eta} = -\eta r + \eta \delta \quad (\text{A. 3})$$

Differentiating (A. 2) logarithmically,

$$\frac{\dot{\eta}}{\eta} = -\sigma \frac{\dot{c}}{c}. \quad (\text{A. 4})$$

⁴⁸It is being assumed that the optimal path is interior. There is no loss of generality here, since the assumption is fulfilled in a steady state. Also, the model permits the household to bring down the capital stock by consuming it away. In steady state, no such consumption will actually take place. A constant fraction of the capital stock could be assumed to undergo *physical* depreciation at each t also. However, this does not add any analytical insight and is avoided for notational simplicity.

Writing $g^d = \frac{\dot{c}}{c}$, equations (A. 3) and (A. 4) imply

$$r = \delta + \sigma g^d. \quad (\text{A. 5})$$

Thus,

$$g^d = g^d(r) = \frac{r - \delta}{\sigma}. \quad (\text{A. 6})$$

Q.E.D.

Proof of Proposition 6.

Plugging (15) in (11), the value of k^* is seen to satisfy

$$f'(k^*) = \delta + \frac{n\alpha}{1 - \alpha} \sigma.$$

The solution for the command economy is found by writing the Hamiltonian as

$$\begin{aligned} H(K(t), c(t), \theta(t)) = & L(t) \frac{1}{1 - \sigma} (c(t)^{1 - \sigma} - 1) \\ & + \theta(t) [F(K(t), K(t)^\alpha L) - L(t)c(t)]. \end{aligned} \quad (\text{A. 7})$$

Equation (A. 4) stays unchanged, except for the replacement of η by θ . Equation (A. 3) transforms to

$$\begin{aligned} \dot{\theta} &= -\frac{\partial F}{\partial K} + \delta\theta \\ &= -\theta \left[\alpha \frac{f(k)}{k} + (1 - \alpha)f'(k) \right] + \delta\theta. \end{aligned} \quad (\text{A. 8})$$

As with Proposition 5, it is easily seen that the equilibrium rate of growth g^{**} for the command economy is still given by (15). Equation (A. 8) then implies

$$\begin{aligned}
 (A.9) \quad \left[\alpha \frac{f(k)}{k} + (1 - \alpha) f'(k) \right] &= \sigma \frac{\dot{c}}{c} + \delta \\
 &= \frac{n\alpha}{1 - \alpha} \sigma + \delta. \quad (A.9)
 \end{aligned}$$

Denoting the solution of (A. 9) by k^{**} , and noting that assumption $F_{11} < 0 \Rightarrow \frac{f(k)}{k} > f'(k) \forall k$, the result follows from a comparison of the solution for k^{**} with that for k^* .

Q.E.D.