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AN INDIRECT DERIVATION OF INPUT-OUTPUT COEFFICIENTS

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"We can make several things clearer, but we can't make anything clear."
— F.P. Ramsey

The almost universal adoption of input-out analysis in development planning has in many underdeveloped countries seriously strained available sources of primary data. In certain instances the data restrictions are such that it becomes unrealistic if not impossible to think in terms of compiling interindustry relations from primary sources. It is logical, therefore, to query the prospects of deriving indirectly sets of technical parameters which in some sense approximate actual technology which could be represented by parameters derived directly from primary data if such data were available.

The following discussion consists of three parts (1) a brief description of both the open and closed input-out models; (2) a modification of the basic model to meet certain conditions usually prevailing in underdeveloped countries - the derivation of allocation coefficients and (3) an indirect derivation of input-output coefficients. Here it is assumed that upper and lower limits can be set on both productions and allocation coefficients and that these boundary values form a system of linear restrictions. Given an optimizing criterion linear programming is then used to obtain an optimum feasible set of input-output parameters.

Basic Model.

The input-output method is essentially an attempted application of the theory of general equilibrium to empirical quantitative analysis. The economy is visualised as a combination of a large number of interdependent activities. Each one of these activities involves the purchase of commodities and services originating in other branches of the economy on the one hand, and the production of commodities and services which are sold to and absorbed by other sectors of the economy on the other. More specifically, an economy considered within the frame-work of an input-output system is regarded as made up of 'n' sectors or industries for each of which a homogeneous additive measure of output or activity is assumed available. Each industry or sector requires certain inputs which it acquires from other sectors; it then sells its output to other sectors to meet their input requirements.

The "industry", defined as households, for example, furnishes its output (services) to other industries in return for consumer goods (household inputs). Government may be treated as an industry which makes payments to other sectors of the economy in return for goods or services and which provides services (its output), the costs of which are met principally by tax levies on the other sectors of the economy. Foreign trade may also be treated as an industry whose inputs are exports and whose product or output is imports. These commodity and service flows (transfers) taking place between the various sectors comprising the economy within some specified period of time, say a year, can be conveniently described by an input-output matrix. The allocation of the total output of any one industry among all the others is shown by a sequence of entries along a particular row, while the distribution of all inputs absorbed by any one industry by origin is at the same time represented by a sequence of entries in the appropriate input columns.

The input-output method depends upon two fundamental kinds of relationships: first, the identity that total output of any given industry is absorbed either by itself or by other industries. This identity is conveniently represented by a set of balance equations. Second, the technological relationship that purchases of any sector from any other sector depend, via a production function, on the level of output of the purchasing sector. Equations (1) represent the former of the two relationships-the distribution of each industry's output.

$$- x_{i1} - x_{i2} - \dots + (x_i - x_{ii}) - \dots - x_{in} = 0 \quad (1)$$

$$i = 1,2,\dots, n$$

Where $x_i \geq 0$ denotes the output of the ith industry and $x_{ij} \geq 0$ the amount of product of the ith industry absorbed by the jth industry. It is assumed that all entries are non-negative and that some (to ensure non-triviality) are positive numbers.

The second or technological relationship may be expressed as follows: Ruling out joint production and assuming constant returns to scale, the production function may be written as

$$x_j = f(x_{1j}\ x_{2j}\ \ldots x_{nj}) \qquad j = 1,2,\ldots,n \qquad (2)$$

Where f is homogeneous of degree 1. The distinctive feature of input-output analysis is that it makes not only both of the above assumptions but the far stronger one of fixed coefficients of prodcution; that is, that a certain minimum amount of each input is required per unit of each output.

If a_{ij} is the required minimum amount of the ithe input per unit of the jth output, (2) can be written as:

$$x_j = \text{Min}\left[\frac{x_{1j}}{a_{1j}}, \frac{x_{2j}}{a_{2j}}, \ldots, \frac{x_{nj}}{a_{nj}}\right] \quad (3)$$

$$j = 1,2,\ldots, n$$

Since x_j equals the smallest of x_{1j}/a_{1j}, x_{2j}/a_{2j}, ..., x_{nj}/a_{nj} it follows that

$$x_j \leq x_{ij}/a_{ij} \qquad i,j = 1,2,\ldots, n \qquad (4)$$

and since no more than the limitational amount of any input will be used the equality sign will hold and (4) can be written as:

$$x_{ij} = a_{ij}x_j \qquad i,j, = 1,2,\ldots, n \qquad (5)$$

The quantity of each input consumed by a sector is therefore directly proportional to the quantity of output produced by it. And the system can be represented by the following set of linear homogeneous equations

$$- a_{i1} x_1 - a_{i2} x_2 - \ldots + (x_i - a_{ii} x_i) - \ldots - a_{in} x_n = 0 \qquad (6)$$

$$i = 1,2,\ldots, n$$

Since the number of equations in (6) equals the number of unknowns a necessary and sufficient condition for a non-trivial solution is that the determinant of the coefficient matrik vanish, that is, $/A/ = 0$.

This describes a so-called closed input-output system. The consumption of all sectors is explained within the model in the course of determining the levels of output of the household, government, and other sectors which form part of a mutually consistent pattern of sector output levels. The open input-output system treats the coefficients in (6) as technological data but assumes that some of the outputs are determined "outside the model," leaving the others to be calculated. In such calculations, a known set of coefficients and a partial list of outputs is used to predict the remaining outputs.

A system is said to be open with respect to consumer demand for example if it does not contain equations describing the structural characteristics of the household sector, or it may be considered open with respect to investment demand which implies that the structural relationships determining investment requirements of all the individual sectors of the economy are not included in the system.

Thus reduced, the number of available equations becomes insufficient to determine uniquely the magnitude of all the unknown variables. However, if arbitary magnitudes are prescribed to some of these values, the corresponding magnitude of all other variables can be determined on the basis of the still available equations. That is to say in an open system it is possible to fix a certain number of variables by deliberate choice while the remaining will fall in accordance with the existing necessities of the still inviolate structural relationships.

Consider a system of m (independent) equations in n variables. With $n > m$. If the values of any n-m of the variables are fixed the remaining m can be solved for. For example, in equation system (6) choose the last sector as one whose final demand is to be fixed and eliminate the last equation. The resulting set of non homogeneous equations is as follows:

$$- a_{i1} x_1 - a_{i2} x_2 - a_{i3} x_3 - \ldots + (1-a_{ii}) x_i - \ldots - a_{im} x_m = a_{in} x_n \quad (7)$$

$$i = 1,2,\ldots, m$$

where $m = (n-1)$

By virtue of $x_{ij} = a_{ij} x_j$ substitute $x_{1n}, x_{2n}, \ldots, x_{mn}$ for $a_{1n} x_n, a_{2n} x_n, \ldots, a_{mn} x_n$ respectively.

$$- a_{i1} x_1 - a_{i2} x_2 - \ldots + (1-a_{ii}) x_i - \ldots - a_{im} x_m = x_{in} \quad (8)$$

$$i = 1,2,\ldots, m$$

Where the x's are a fixed "bill of goods" in terms of which the system of equations can be solved.

Reduced to matrix form the balance equations can be expressed as:

$$x = Xi + f \qquad (8)$$

Where i is a unit column vector with m elements, Xi a vector of intermediate outputs obtained by adding up the flaws in every row, and f a vector of final demand. The assumption that inputs are proportional to inputs is expressed as:

$$X = A\hat{x} \quad (9)$$

Where A a matrix of a_{ij}'s is of order m and $\hat{x}$ a diagonal matrix whose elements are the elements of X. Thus:

$$\begin{aligned} x &= AXi + f \\ &= Ax + f \\ &= (I-A)^{-1}f \qquad (10) \end{aligned}$$

Where I is the unit matrix of order m. This yields the open model connecting the total output vectors x with the vectors of final demand, f, by means of the matrix multiplier $(I-A)^{-1}$.

Allocation Coefficients.

It has been assumed that inputs are constant proportions of outputs and that there are no substitutions in the economy. This implies the further assumption that what ever "final bill of goods" may be required, the production coefficients are immutable and determine the allocation of outputs. This is justified, even if the system allows for alternative combinations, providing scarcities do not appear in any of the industries. Assuming that alternative combinations are available as scarcities arise there will be attempts at rationing the scarce commodities and encouraging substitution for them. Thus an input-output transaction matrix may be conceived in terms of an equilibrium position of two sets of interacting forces; one set as the technical factors expressed through production functions, the other set as market factors expressed through allocation functions.

In a competitive market where resources are not in short supply the production as opposed to the allocation coefficients will determine the point of equilibrium. In a monopolistic market however, where resources are in short supply the allocation and not the production functions will determine which of alternative processes or input combinations will be chosen by any particular industry.

Thus we can associate with an input-output matrix two sets of coefficients; one consisting of the production of technical coefficients,

$$A = X\hat{x}^{-1}$$

The other the allocation coefficients expressed by the relationship

$$A^{*} = \hat{x}^{-1}X \qquad (11)$$

Reality may be better approximated by one or the other of these relationships according to whether supply or demand influences predominate.

The original Leontief formulation assumes the A matrix fixed but allows A* to vary with changes in the final bill of demand. This presupposes that there are no scarcities so that the flows or intermediate products are determined by demand. Thus even in the short run there must be unused capacity in each industry so that a change in final demand does not alter relative prices or introduce shortages. But excess capacity is not evenly distributed over the different industries and this tends to restrict the free flow of supplies in response to changes in the composition of final demand.

Consider an economy under some form of central control and with insufficient resources and capacities in most industries in relation to the targets aimed at. In such an economy certain technical combinations may be dictated not by normal requirements of the production process but by scarcity and consequent rationing. Input ratios are conditioned by the assigned quotas and any change in the assigned quota may lead to alterations in the input coefficients. In such a situation the stability of the production coefficients can not be assumed for changes in final demand. It may in fact be that allocation coefficients tend to be more stable over a short period than the corresponding technical coefficients since rationing authorities, once the relative shares of each industry have been decided may resist change due to a delicate balance of claims and counter claims.

An allocation model may be described as follows: A new coefficient matrix is defined as:

$$A^* = \hat{x}^{-1} X$$

such that $$A^* i \leq i$$

The equality sign holding only in the case of a closed model for which we have the relationship

$$x = X'i$$

transposing A* and substituting

$$A^{*'} = X'\hat{x}^{-1}$$

$$x = A^{*'}\hat{x}i$$

$$x = A^{*'}\hat{x} \qquad (12)$$

Equation (12) states that the input from ith industry into the jth industry is a constant proportion of the output of the ith industry so that changes in this output will be allocated in fixed proportions to each of the other industries. Moving to the open model and considering factor incomes, y, as exogenous the following relations are obtained:

$$x = A^{*'}x + y$$

$$(I-A^{*'})x = y$$

$$x = (I-A^{*'})y \qquad (13)$$

Where y is a column vector of factor costs. It is understood that x now has one less row and A*' one less row and one less column than in the closed model of (12). (y must be of the same dimension as x).

Estimating Input-Output Coefficients Under Conditions of Limited Data.

Since the input-output method offers certain advantages in development planning i.e., determining consistency of a plan, choosing among various investment projects etc., there is reason to consider the problem of estimating the technical parameters on the basis of limited data since sufficient and accurate primary data are typically not available in many underdeveloped countries. One method of deriving estimates of the desired coefficients where data is limited is to assume that upper and lower limits can be set on both production and allocation coefficients and then minimise the error between projected and observed output levels. This leads to a linear programming problem that can be demonstrated as follows.

For any given output vector x the matrix of production coefficients, A, and the matrix of allocation coefficients, A*, will be related by the transformation

$$A^* = \hat{x}^{-1} A \hat{x} \qquad (14)$$

since $$X = A\hat{x} = \hat{x}A^*$$

Now consider two matrices, L and U, whose elements are lower and upper limits of the elements of A. Then

$$L \leq A \leq U \qquad (15)$$

Where all the elements of U are at least as large as the corresponding elements of A, which in turn are at least as large as the corresponding elements of L. Similarly

$$L^* \leq A^* \leq U^* \qquad (16)$$

Assuming that L in (15) holds for some given output vector

$$A^*_L = \hat{x}^{-1} L \hat{x} \qquad (17)$$

if U holds

$$A^*_u = \hat{x}^{-1} U \hat{x} \qquad (18)$$

thus, given x, if A is forced to one of its limits an associated allocation matrix is determined. There is no assurance however that either A^*_L or A^*_U will satisfy the inequality $L^* \leq A^* \leq U^*$.

Denote by AQ the matrix of estimated production coefficients and assume that upper and lower limits of A and A* are determined exogenously by planning authorities, engineering relations etc. then

$$L \leq AQ \leq U \qquad (19)$$

$$L^* \leq AQ^* \leq U^* \qquad (20)$$

If x_Q were to be produced with an arbitrarily defined set of technological relations denoted by A_o the corresponding allocation matrix would be

$$A_Q^* = \hat{x}_Q^{-1} A_o \hat{x}_Q \qquad (21)$$

Assuming that A_Q^* exceeds either L^* or U^* A_o would necessarily have to be altered within the limits L and U in such a way that the value adopted, A_Q, gives

$$L^* \leq \hat{x}^{-1} A_Q \hat{x} \leq U^* \qquad (22)$$

the problem is then to find a value of A_Q subject to the following conditions:

$$L \leq A_Q \leq U \qquad (23)$$

$$L^* \leq A_Q^* \leq U^* \qquad (24)$$

$$x_Q = A_Q x_Q = A_Q^{*'} x_Q \qquad (25)$$

(Since in a closed system $x = x'i$) and

$$A'i = A^*i = i \qquad (26)$$

This represents a form of linear programming problem except that since there is no optimising criterion only solutions which are feasible can be obtained. Before a linear programming algorythum can be used in determining A_Q it is necessary to define an optimising criterion-objective function-which will permit the coefficient matrices to be modified as output levels are subjected to changes.

Assume for some base year that actual output levels for the country considered can be observed and form the elements of a vector x. If then true technology is approximated by A_Q it follows that the difference between x and x_Q, the output vector associated with A_Q, will be minimum. An objective function might therefore be defined as follows:

$$Z \min = \sum_{i=1}^{n} (x_i - x_{iQ})$$

$$= \sum_{i=1}^{n} (x_i - \sum_{j=1}^{n} x_{ij})$$

since $x_i = \sum_{j=1}^{n} x_{ij}$

$$= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \qquad (27)$$

Z is then minimised subject to the following constraints:

$$(a_{ij}^{\ell} x_j) \leq x_{ij} \leq (a_{ij}^{u} x_j) \qquad (28)$$

$i = 1,2,\ldots, n$

$j = 1,2,\ldots, n$

$$(a_{ij}^{*\ell} x_i) \leq x_{ij} \leq (a_{ij}^{*u} x_j) \qquad (29)$$

$i = 1,2,\ldots, n$

$j = 1,2,\ldots, n$

since $a_{ij} = \frac{x_{ij}}{x_j}$, $a^{*}_{ij} = \frac{x_{ij}}{x_i}$

and $\sum_{j=1}^{n} x_{j1} = \sum_{i=1}^{n} x_{1i} = x_1$

$$\sum_{j=1}^{n} x_{j2} = \sum_{i=1}^{n} x_{2i} = x_2 \qquad (30)$$

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$$\sum_{j=1}^{n} x_{jn} = \sum_{i=1}^{n} x_{ni} = x_n$$

$$x_{ij} \geq 0 \qquad (31)$$

The equations in (30) express the fact that for each industry total costs equal total sales equal the value of total output. The inequalities in (28) and (29) have been formed by taking upper limits for A and A* and multiplying them by the output vector. The limits chosen need not in general be symmetrical.

The relations (27) - (30) plus the non negativity restrictions in (31) structure the problem such that a linear optimising algorythum can be used to determine an optimum feasible set of input-output parameters which comprise AQ. It should be emphasised that this method is purely exploratory. There is absolute dependence on a priori determined upper and lower limits in the production and allocation coefficients. The difficulty, of course, is how to determine these limits in advance. The results ought, therefore, to be taken as nothing more than an exercise in method suggesting possible future research.